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FINAL REPORT
Robust Control System Design
AFOSR Grant 91-0036
1 Oct. 1990 – 30 Sept. 1995

Principal Investigator
J.B. Pearson
Rice University
P.O. Box 1892
Houston, Texas 77251-1892

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DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

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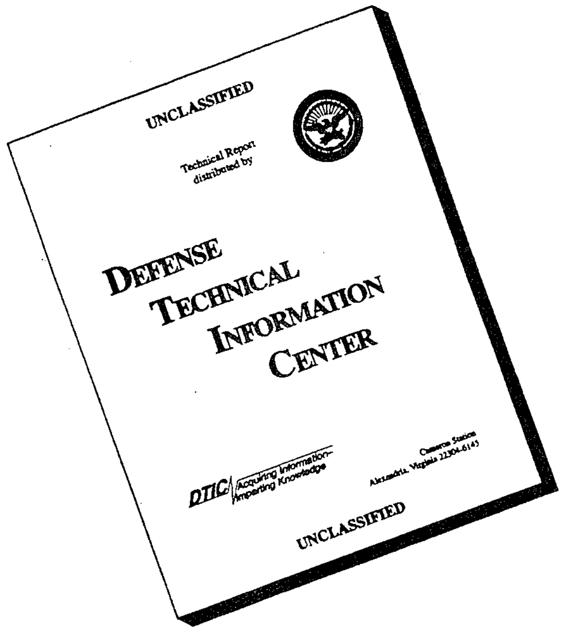

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This research was concerned with the robust optimal control of systems subjected to structured uncertainty. The design objective was to minimize the induced system norm (i.e. maximum gain) in the cases of ℓ_2/L_2 and ℓ_∞/L_∞ inputs. Major results were obtained in the case of linear discrete-time systems with nonlinear/time-varying uncertainty and for continuous systems controlled by digital computers (sampled-data systems). The sampled-data results are now a part of the MATLAB μ-tools toolbox and the structured uncertainty results for discrete-time systems has led to an efficient "D-K-type" synthesis procedure for minimizing the ℓ_∞ induced system norm.

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1. Research Objectives

The objectives of this research were to develop theory and computational methods to formulate and solve robust control system design problems. The initial stages of the research were devoted to the design objective of minimizing the maximum peak gain of a system containing structured nonlinear/time-varying uncertainty. Next, the focus was on continuous-time systems controlled by digital computers, i.e. sampled-data systems. This work was motivated by the fact that continuous-time compensators for continuous-time plants are irrational when the design objective is minimizing peak-gain. Therefore it was necessary to study the sampled-data problem and develop new methods for its solution.

2. Status of Research Effort

The objectives in this research have been achieved in a most satisfactory manner. Two of the most significant results were 1) the work of Mustafa Khammash on performance robustness in the presence of structured uncertainty and 2) the work of Bassam Bamieh on sampled-data control.

Briefly, these results are as follows:

One of the most significant developments in H_∞ control system design was establishing the equivalence of the robust performance problem and a robust stability problem with structured perturbations. This is generally attributed to John Doyle and made possible the " μ-synthesis" approach to control system design. In this approach, the design objective is to minimize the maximum error energy. When the objective is to minimize the maximum error magnitude, the problem becomes quite different and more complicated.

Mustafa Khammash in his Ph.D. thesis established an equivalence between a robust performance problem and a robust stability problem for discrete-time systems in the presence of time-varying/nonlinear structured uncertainty. This was a major breakthrough in the solution of the maximum magnitude problem, which has been referred to as the ℓ_1 optimal control problem. In his further work, he developed an iterative technique similar to μ-synthesis in order to design optimal controllers. In the case of discrete-time systems, the procedure is much simpler since the scaling factors are constant, rather than functions of frequency, and the optimum can be calculated exactly at each step.

Khammash's results can be applied to the analysis of continuous-time systems, but not to the synthesis of optimal continuous-time controllers. The earlier work of Munther Dahleh, also done here at Rice University, had shown that in general, optimal continuous-time problems had irrational solutions and the realization of these solutions posed many problems. As a result, a different approach to the problem was proposed, in that a continuous-time plant would be controlled by a digital (discrete-time) device. The accepted terminology for such systems is "sampled-data-systems,"

and the objective was to minimize the maximum value of the continuous-time system error for such systems.

The Ph.D. thesis of Bassam Bamieh presented a solution to both an H_∞ version and an L_1 version of this problem. Bamieh developed a procedure called "lifting," by which a periodically sampled system could be converted to a discrete-time system in which the induced system norms were the same (i.e. the system gains). This meant that the maximum value of the continuous-time system error is equal to the maximum value of the discrete-time system error. This equivalence is also true for any induced norm, in particular for the H_∞ and L_1 norms.

Bamieh worked out the details of the transformation which gives an exact solution to the H_∞ problem and an approximate solution to the L_1 problem. Programs that implement his H_∞ solution are now incorporated into MATLAB μ-tools.

Recent work has involved the study of robust solutions to tracking problems. In particular, the object is to design compensators to minimize the maximum "steady-state" errors in sampled-data systems with structured uncertainty. This is quite complicated, in general, and the initial work deals with discrete-time systems with structured uncertainty. The foundations for solving such problems have been laid by Mustafa Khammash and analysis of discrete-time systems with structured uncertainty is straightforward. Synthesis is more difficult. Algorithms have been developed for solving certain problems involving scalar plants with two structured perturbations. At the present time, the general problem seems to be a least as difficult as the "μ–synthesis" problem and a different framework may lead to more reasonable computations. The search for such a framework will be a continuing goal for future studies.

3. Journal Publications

(a) M. Khammash and J.B. Pearson, "Performance robustness of discrete-time systems with structured uncertainty," *IEEE Trans. Auto-Control*, Vol. 36, No. 4, pp. 398-412, 1991.

(b) B. Bamieh and J.B. Pearson, "A general framework for linear periodic systems with application to H^∞ sampled/data control," *IEEE Trans. Auto-Control*, Vol. 37, No. 4, pp. 418-435, April, 1992.

(c) B. Bamieh, J.B. Pearson, B.A. Francis, and A. Tannenbaum, "A lifting technique for linear periodic systems with applications to sampled-data control," *Systems and Control Letters*, Vol. 17, pp. 79-88, 1991.

(d) B. Bamieh, M.A. Dahleh, and J.B. Pearson, "Minimization of the L^∞-induced norm for sampled data systems," *IEEE Trans. Auto Control* Vol. 38, No. 5, pp. 717-732, May 1993.

(e) B. Bamieh and J.B. Pearson, "The H^2 Problem for sampled-data systems," *Systems and Control Letters*, Vol. 19, pp. 1-12, 1992.

(f) M. Khammash and J.B. Pearson, "Analysis and design for robust performance with structured uncertainty," *Systems & Control Letters*, 20, (1993), pp. 179-187.

(g) A.P. Kishore and J.B. Pearson, "Kernel representation and properties of discrete-time input-output systems," *Linear Algebra and Its Applications*, Special Issue on Linear Systems Theory, Vols. 205-206, July 1994..

4. Personnel

The following graduate students were associated with the research effort and received their degrees as shown:

(a) Mustafa Khammash, Ph.D., May 1990
"Stability and Performance Robustness of Discrete-time Systems with Structured Uncertainty"

(b) Bassam Bamieh, Ph.D., August 1991
"Analysis and Robust Control of Hybrid Continuous/Discrete Time Systems"

(c) James S. McDonald, Ph.D., August 1992
"Design of Control Systems to meet ℓ_∞ Specifications"

(d) Ananda Kishore, Ph.D., May 1995 "Studies in System Representation and Control"

(e) Behnam Sadeghi, to receive M.S., May 1996
Tentative thesis title: "Controller Design to Minimize Maximum Tracking Errors"

5. Conference Presentations

(a) M. Khammash and J.B. Pearson, "Robust Disturbance Rejection in l^1 Optimal Control Systems," *Proceedings of the 1990 American Control Conference*, San Diego, CA, Vol. 1, pp. 945-951, May 1990.

(b) M. Khammash and J.B. Pearson, "Performance Robustness of Discrete-time Systems with Structured Uncertainty," *Proceedings of the 1990 Conference on Decision and Control*, Honolulu, Hawaii, pp. 414-420, December 1990.

(c) M. Khammash and J.B. Pearson, "Robustness in the Presence of Structured Uncertainty," *Proceedings of the 1991 MTNS Conference*, Kobe, Japan, June 1991.

(d) M. Khammash and J.B. Pearson, "Robustness Synthesis for Discrete-Time Systems with Structured Uncertainty," *Proceedings of the 1991 American Control Conference*, Boston, MA, June 1991.

(e) B. Bamieh and J.B. Pearson, "The H^2 Problem for Sampled-Data Systems," *Proceedings of the 1991 Conference on Decision and Control*, Brighton, England, December 1991.

(f) B. Bamieh, M.A. Dahleh and J.B. Pearson, "Minimization of the L^{∞}-Induced Norm for Sampled-Data Systems," *Proceedings of the 1992 American Control Conference*, Chicago, IL., June 1992.

(g) J.S. McDonald and J.B. Pearson, "Formulation of ℓ_1-Optimal Control Problems without Interpolation," *Proceedings of the 1992 American Control Conference*, Chicago, IL., June 1992.

(h) A.P. Kishore and J.B. Pearson, "Uniform Stability and Performance in H_{∞}" *Proceedings of the 31st IEEE Conference on Decision and Control*, December 16-18, 1992, Tucson, Arizona.

(i) J.S. McDonald and J.B. Pearson, "Control System Design to meet Weighted l_{∞} Specifications," *Proceedings of the 1993 IFAC Congress*, 18-23 July 1993, Sydney, Australia.

(j) A.P. Kishore and J.B. Pearson, "Behavior is More Fundamental than Representations," Proceedings 32nd IEEE Conference on Decision and Control, San Antonio, TX Dec. 15-17, 1993.

(k) J.B. Pearson, "Linear Multivariable Control and the Development of L_1 Optimal Control," Invited paper in a Special Session entitled "Historical and Fundamental Developments in Control Systems." 33rd IEEE CDC, Lake Buena Vita, Florida, Dec. 14-16, 1994.

Performance Robustness of Discrete-Time Systems with Structured Uncertainty

Mustafa Khammash and J. Boyd Pearson, Jr., *Fellow, IEEE*

***Abstract*—Given an interconnection of a nominal discrete-time plant and a stabilizing controller together with structured, norm-bounded, nonlinear/time-varying perturbations, necessary and sufficient conditions for robust stability, and performance of the system are provided. This is done by first showing that performance robustness is equivalent to stability robustness in the sense that both problems can be dealt with in the framework of a general stability robustness problem. The resulting stability robustness problem is next shown to be equivalent to a simple algebraic one, the solution of which provides the desired necessary and sufficient conditions for performance/stability robustness. These conditions provide an effective tool for robustness analysis and can be applied to a large class of problems. In particular, it is shown that some known results can be obtained immediately as special cases of these conditions.**

I. Introduction

Obtaining good mathematical models of physical systems is important for their effective control. In general, the better the model, the more one expects from an optimal controller for this system. Ideally, a mathematical model that describes exactly the real system should be obtained. Based on that model, a controller that achieves certain objectives can then be designed. When implemented on the real system, one expects it to achieve the design objectives. However, this rarely takes place in practice for many reasons. First, obtaining an exact model is generally not possible and one must use approximate models. Second, better models tend to be more complicated in order to capture more accurately the dynamics of the system to be controlled, and so despite the availability of a good model, a simpler less accurate one might be used in order to simplify the design and analysis procedures and to make use of those tools for controller design which are based on the simpler but less accurate approximation. An example of this is the linearization of a nonlinear system about an operating point. Third, and equally important, even if the underlying physical system could be modeled accurately at one point in time, parameter variations that could appear for any one of many reasons eventually take their toll on the system and render the model inaccurate. For all these reasons, a controller that achieves good performance when controlling the model, might not perform so well when used to control the actual plant and could even make the system unstable. In short, robustness to model uncertainties is an important objective and should be an integral part of any controller design.

For systems with bounded energy signals, the $\mathscr{H}^\infty$ norm is the most suitable norm to use. When dealing with robust performance in the context of linear feedback systems with $\mathscr{H}^\infty$ norm performance objectives, the paper by Doyle [1] introduces a nonconservative measure of performance for linear feedback systems in the presence of structured model uncertainties [1].[1] This approach is based on a matrix function called the structured singular value, where stability and performance robustness are dealt with in the same framework. The class of perturbations treated are linear time-invariant norm-bounded perturbations.

When the system at hand does not involve bounded energy signals but rather bounded magnitude signals as is the case when bounded persistent disturbances are present, the more suitable norm is the $\mathscr{A}$ norm or l^1 norm. In [2], [3], Dahleh and Pearson provided a complete solution to the problem of minimizing the $\mathscr{A}$ norm of a linear time-invariant continuous/discrete-time system through the choice of a stabilizing controller. The optimal controllers obtained in the discrete-time case are more useful than those in the continuous-time case since they are easier to implement physically.

In this paper, we present a solution to the robustness problem in the l^1 setting. The class of perturbations considered consists of norm bounded perturbations allowed to be time varying or nonlinear. We provide necessary and sufficient conditions for stability robustness for structured perturbations where any number of perturbations can enter between any two points in the system. In addition, we allow performance objectives to be achieved in a robust manner subject to robust stability. This is done by showing that the stability and performance robustness problem is equivalent to a simple algebraic problem which can be easily solved to give the desired nonconservative conditions for stability and performance robustness. We show how the results in [4] and [5] can be obtained as special cases of this theory. Finally, we provide some examples demonstrating how a controller that achieves robust stability and performance can be designed.

The paper is divided into nine sections. Section II introduces the notation, while Section III provides some preliminary results. In Section IV, the stability and performance robustness problem is set up. In Section V, we prove Theorem 1 which establishes that a performance robustness problem is in fact equivalent to a stability robustness problem when the perturbations are linear, time-varying, and norm bounded. In Section VI, we show that the stability robustness problem is equivalent to an algebraic problem which gives us the means by which to obtain necessary and sufficient conditions for stability robustness and consequently performance robustness. In Section VII, the results are extended to include nonlinear norm-bounded perturbations. In Section VIII, some applications of the theory are provided, and finally, Section IX contains some concluding remarks.

II. Notation

$\mathbb{R}^q$ The space of q-tuples of real numbers. If $x = (x_1, \cdots, x_q) \in \mathbb{R}^q$, then $|x|_\infty := \max_i |x_i|$.

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The authors are with the Department of Electrical and Computer Engineering, Rice University, Houston, TX 77251-1892.

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[1] See also [12].

$\mathbb{R}^+$ Nonnegative real numbers.

$\mathscr{M}(p \times q)$ Set of real matrices having p rows and q columns.

l^∞ Space of all bounded sequences of real numbers, i.e., $x = \{x(k)\}_{k=0}^{\infty} \in l^\infty$ if and only if $\sup_k |x(k)| < \infty$. If $x \in l^\infty$ then $\|x\|_\infty = \sup_k |x(k)|$.

l_q^∞ Space of q-tuples of elements of l^∞. If $x = (x_1, \cdots, x_q) \in l_q^\infty$ then $\|x\|_\infty = \max_i \|x_i\|_\infty$.

$l_{q,e}^\infty$ Extended l_q^∞ space. It is equal to the space of all q-tuples of sequences of real numbers.

π_i If $x = (x_1, \cdots, x_q) \in l_{q,e}^\infty$, then $\pi_i x := x_i \in l_e^\infty$.

l^1 Space of absolutely summable sequences. If $x \in l^1$ then $\|x\|_1 = \sum_{k=0}^{\infty} |x(k)| < \infty$.

$l_{p\times q}^1$ Space of $p \times q$ matrices with entries in l^1. If $x = (x_{ij}) \in l_{p\times q}^1$, then $\|x\|_1 := \max_{1\le i\le p} \sum_{j=1}^{q} \|x_{ij}\|_1$.

P_k The truncation operator on sequences. Hence if $x = \{x(i)\}_{i=0}^{\infty}$ is any sequence, then $P_k x = \{x(0), x(1), \cdots, x(k), 0, \cdots\}$.

$P_N(l_q^\infty)$ The set of all $x \in l_q^\infty$ such that $x_i(k) = 0 \; \forall k > N$ and $1 \le i \le q$.

S_k Right shift by k positions. If $x = \{x(i)\}_{i=0}^{\infty}$ is any sequence and k is a nonnegative integer, then $S_k x = \{\overbrace{0, \cdots, 0}^{k}, x(0), x(1), \cdots\}$. On the other hand, $S_{-k} x = \{x(k), x(k+1), \cdots\}$. Hence $S_{-k} S_k = I$ but $S_k S_{-k} \neq I$.

$\mathscr{L}_{TV}^{p\times q}$ The space of all bounded linear causal operators mapping l_q^∞ to l_p^∞. If $R \in \mathscr{L}_{TV}^{p\times q}$ then $\|R\| := \sup_{x\neq 0} \|Rx\|_\infty / \|x\|_\infty$ which is the induced operator norm. Each R in $\mathscr{L}_{TV}^{p\times q}$ can be completely characterized by its block lower-triangular pulse response matrix.

$\mathscr{L}_{TI}^{p\times q}$ Subspace of $\mathscr{L}_{TV}^{p\times q}$ consisting of time-invariant operators. For each $R \in \mathscr{L}_{TI}^{p\times q}$ corresponds a unique r in $l_{p\times q}^1$ where r_{ij} is the impulse response of R_{ij}, the component of R mapping the jth input to the ith output.

$\left[\frac{x}{y}\right]$ If $x \in \mathbb{R}^p$ and $0 \neq y = (y_1, \cdots, y_q) \in \mathbb{R}^q$ and if $i_{\max}$ is the smallest indexing integer such that $|y_{i\max}| \ge |y_i|$ for $i = 1, \cdots, q$ then $[x/y]$ is defined to be the real $p \times q$ matrix formed by setting its $i_{\max}$th column to be $(1/y_{i\max})\, x$ and all of the other columns to zero. A consequence of this definition is that $[x/y]y = x$.

III. Preliminaries

For the sake of completeness and in order to establish notation, we review in this section some of the concepts pertaining to feedback systems. Let G: $l_{p,e}^\infty \to l_{q,e}^\infty$ be any map. G is said to be *causal* if $P_k G = P_k G P_k$ for all $k \ge 0$. It is said to be *strictly causal* if $P_k G = P_k G P_{k-1}$ for all $k \ge 0$.

Consider the feedback interconnection in Fig. 1, where G: $l_{p,e}^\infty \to l_{q,e}^\infty$ and H: $l_{q,e}^\infty \to l_{p,e}^\infty$ are both causal maps. The system depicted is said to be *well-posed* if $(I - GH)^{-1}$ exists as a map from $l_{q,e}^\infty$ to $l_{q,e}^\infty$, and is causal. It is said to be *l^∞-stable* iff

1) it is well-posed,

2) the map $(u_1, u_2) \mapsto (e_1, e_2, y_1, y_2)$ takes $l_p^\infty \times l_q^\infty$ into $l_p^\infty \times l_q^\infty \times l_p^\infty \times l_q^\infty$,

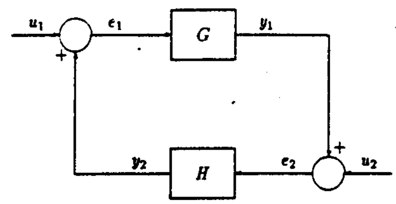


Fig. 1. Feedback system.

3) there exists real numbers α_1 and α_2, independent of u_1 and u_2, such that for all u_1 and u_2

$$\|e_1\|_\infty, \|e_2\|_\infty, \|y_1\|_\infty, \|y_2\|_\infty \le \alpha_1 \|u_1\|_\infty + \alpha_2 \|u_2\|_\infty.$$

A map G: $l_{p,e}^\infty \to l_{q,e}^\infty$ is said to be l^∞-stable if it is causal, takes l_p^∞ into l_q^∞, and is bounded, i.e., there exists $\alpha > 0$ such that $\|Gu\|_\infty \le \alpha \|u\|_\infty$ for all $u \in l_p^\infty$. Clearly, if G and H in Fig. 1 are both l^∞-stable and if the system is well-posed then a necessary and sufficient condition for the system in the figure to be l^∞-stable is that $(I - GH)^{-1}$ and $(I - HG)^{-1}$ are both l^∞-stable. In fact, as the next proposition shows, it is enough to check that only one of them is stable; the other will follow suit.

Proposition 1: Let G: $l_p^\infty \to l_q^\infty$ and H: $l_q^\infty \to l_p^\infty$ both be l^∞-stable maps. Then $(I - GH)^{-1}$ is l^∞-stable if and only if $(I - HG)^{-1}$ is l^∞-stable.

Proof: ($\Rightarrow$) Assume $(I - GH)^{-1}$ is l^∞-stable. It may be easily verified that $(I - HG)^{-1} = I + H(I - GH)^{-1}G$, which is l^∞-stable. The other direction of the proof is identical. ∎

If $A = (a_{ij}) \in \mathscr{M}(p \times q)$, then the induced-operator norm of A as a map from $(\mathbb{R}^q, |\cdot|_\infty)$ to $(\mathbb{R}^p, |\cdot|_\infty)$ is defined by

$$|A|_\infty := \sup_{|x|_\infty \le 1} |Ax|_\infty = \max_{1\le i\le p} \sum_{j=1}^{q} |a_{ij}|.$$

We use this to give an expression for the norm of an element $R \in \mathscr{L}_{TV}^{p\times q}$. R can be completely characterized by its pulse response matrix which has the following form:

$$\begin{pmatrix} R_{00} & & 0 \\ R_{10} & R_{11} & \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where $R_{ij} \in \mathscr{M}(p \times q)$. This infinite matrix representation of R acts on elements of l_q^∞ by multiplication, i.e., if $u \in l_q^\infty$, then $y := Ru \in l_p^\infty$ where $y(k) = \sum_{j=0}^{k} R_{kj} u(j) \in \mathbb{R}^p$. It can now be seen that the induced-operator norm of R is equal to $\sup_i |(R_{i0} \cdots R_{ii})|_\infty$.

When restricted to $\mathscr{L}_{TI}^{p\times q}$, the time-invariant subspace of $\mathscr{L}_{TV}^{p\times q}$, another representation of the elements of $\mathscr{L}_{TI}^{p\times q}$ is more convenient. This alternate representation results from the fact that corresponding to each $R \in \mathscr{L}_{TI}^{p\times q}$, there is an element $r = (r_{ij}) \in l_{p\times q}^1$ such that r_{ij} is the pulse response of that component of R mapping the jth input to the ith output. In this case, the induced-operator norm of R as a map from l_q^∞ to l_p^∞ is equal to the norm of r in $l_{p\times q}^1$, which we shall also refer to as the $\mathscr{A}$ norm. Hence, $\mathscr{L}_{TI}^{p\times q}$ is isomorphic to $l_{p\times q}^1$, and each operator in $\mathscr{L}_{TI}^{p\times q}$ is uniquely determined by its pulse response in $l_{p\times q}^1$ whose norm will be equal to norm of the operator in $\mathscr{L}_{TV}^{p\times q}$.

IV. Problem Setup

We are mainly interested in l^∞ signals and discrete-time systems. Aside from that, the only conditions imposed will be those needed to guarantee the well-posedness of the problem.

Common to all the problems in which stability and performance of a certain system are to be studied under the effect of perturbations are a nominal plant and a controller stabilizing it. In our case, both of these are assumed to be linear time-invariant discrete-time systems. There is no reason why only one nominal plant or controller can be considered, and so, as many as desired can be incorporated as long as the resulting nominal system is stable. As for the perturbations, they are first modeled as strictly causal linear maps taking l^∞ signals to l^∞ signals with bounded-induced norms. Hence, the perturbations are allowed to be time varying. Nonlinear perturbations are treated in Section VII. There can be as many perturbations as desired and they can enter anywhere in the system. So for a specific set of bounds on the norms of the perturbations, we have a family of systems each of which is composed of the nominal part and a set of fixed perturbations with norms less than the corresponding given bounds. The first objective is to determine when every member of that class of systems is stable, i.e., when our system is robustly stable. In many cases, stability is not all that is required from a system and certain performance objectives are to be met. A useful and popular objective is keeping small the norm of the function mapping an external input, say u, to a certain signal in the loop, call it y. Since there could be more than one such objective, let us denote the resulting functions by $T_{y_i u_i}$ for $i = 1, \cdots, m$, where $T_{y_i u_i}$ is the function mapping signals at point u_i to signals at point y_i. Because we are mainly concerned with l^∞ signals, the norm we want to be small would be in our case the induced l^∞ norm. Now our objective is to determine, given a set of m positive real numbers $\gamma_1, \cdots, \gamma_m$, conditions under which our system is stable and satisfies $\| T_{y_i u_i} \| < \gamma_i$ for all allowable perturbations. In other words, when does our system achieve robust performance?

We now formally set up the stability and performance robustness problem mentioned previously. The configuration we shall use in the setup of the robustness problem is shown in Fig. 2. In the figure, M represents the interconnection of the nominal plant and the stabilizing controller, and is therefore linear, time invariant, and stable. Each Δ_i represents the perturbations between two points in the system, and has norm less than or equal to one. Of course, there is no loss of generality in assuming that the chosen bound on the norms of each of the Δ_i's is one, since any other set of numbers could be absorbed in M. We will restrict the Δ_i's to be *strictly* causal in order to guarantee the well-posedness of the system. This is not a serious restriction and can be removed if it is known that the perturbation- nominal-system connection is well-posed. Accordingly, we can define the classes of perturbations to which the Δ_i's belong. Assuming the perturbations enter at n places, and that each has p_i inputs and q_i outputs we have

$$\Delta_i \in \Delta(p_i, q_i)$$

where $\Delta(p_i, q_i) := \{\Delta \in \mathscr{L}_{TV}^{p_i \times q_i} |$

$$\Delta \text{ is strictly causal and } \|\Delta\| \leq 1\}$$

for $i = 1, \cdots, n$.

Note that Δ_i is not dependent in any way on Δ_j when $j \neq i$. The only restriction is that Δ_i belongs to $\Delta(p_i, q_i)$ for each i. Next, let $p = \sum_i p_i$, and $q = \sum_i q_i$. By $\mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$ we mean the set of all operators mapping l_q^∞ to l_p^∞ of

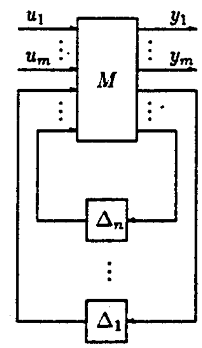


Fig. 2. Stability and performance robustness problem.

the following form:

$$D = \begin{pmatrix} \Delta_1 & 0 & \cdots & 0 \\ 0 & \Delta_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \Delta_n \end{pmatrix}$$

where Δ_i belongs to $\Delta(p_i, q_i)$. When the pairs (p_i, q_i) are known, they will be dropped from the notation and $\mathscr{D}$ will be understood to mean the aforementioned set. We will say the system in Fig. 2 achieves robust stability if the system is stable for all $D \in \mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$. We will say it achieves robust performance if it achieves robust stability and $\| T_{y_i u_i} \| < 1$ for all i and for all D in $\mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$.

In the context of this setup, our problem can be stated as follows.

Problem Statement: Find necessary and sufficient conditions for the system in Fig. 2 to achieve robust performance.

V. Performance Robustness Versus Stability Robustness

In this section, we will establish a useful relationship between stability and performance robustness, that will be used later in the solution of our problem. This is achieved in Theorem 1 which is the main result in this section. To aid in the proof of this theorem, we will need to determine necessary and sufficient conditions that a linear time-varying operator $R \in \mathscr{L}_{TV}^{q \times p}$ must satisfy in order for $(I - R\Delta)^{-1}$ to be l^∞-stable for all $\Delta \in \Delta(p, q)$. Such conditions are provided by Lemma 1, to be stated next. However, the conditions given in this lemma are somewhat nonintuitive in the sense that they do not readily translate into conditions on R. By utilizing Lemma 2, Lemma 3 restates Lemma 1 in a form that relates to R more closely and thus takes care of this shortcoming. Finally, we note that the sufficiency part in Lemma 3 has been proven in [6]. Nevertheless, since the extra effort required to reprove it using the techniques of this paper is minimal, we prove it again here and provide a proof for the necessity.

Lemma 1: Let $R \in \mathscr{L}_{TV}^{q \times p}$. Then $(I - R\Delta)^{-1}$ is not l^∞-stable for some $\Delta \in \Delta(p, q)$ if and only if there exists a real number $c > 0$ and $\xi \in l_{p,e}^\infty \setminus l_p^\infty$ such that

$$\| P_k \xi \|_\infty \leq \| P_{k-1} R \xi \|_\infty + c \qquad \forall k \geq 0.$$

The proof of this lemma is postponed until Section VI.

Lemma 2: Let $R \in \mathscr{L}_{TV}^{q \times p}$, and $\xi \in P_N(l_p^\infty)$. Then given $\epsilon > 0$ and $a > \| \xi \|_\infty$, there exist an integer $\bar{N} > N$ and $\bar{\xi} \in P_{\bar{N}}(l_p^\infty)$

such that

$$P_N \tilde{\xi} = \xi \qquad (\tilde{\xi} \text{ is a truncated extension of } \xi)$$
$$\|\tilde{\xi}\|_\infty = a.$$
$$\|P_{\tilde{N}} R \tilde{\xi}\|_\infty \ge \|S_{-(N+1)} R S_{(N+1)}\| \cdot a - \epsilon.$$

Proof: From the pulse response matrix representation of R it can be seen that

$$\|S_{-(N+1)} R S_{(N+1)}\| = \sup_{i \ge 1} |(R_{N+i,N+1} \cdots R_{N+i,N+i})|_\infty.$$

It follows that for some $\tilde{N} > N$, $\tilde{R} := (R_{\tilde{N},N+1} \cdots R_{\tilde{N},\tilde{N}}) \in \mathscr{M}(q \times p(\tilde{N} - N))$ satisfies

$$|\tilde{R}|_\infty \ge \|S_{-(N+1)} R S_{(N+1)}\| - \frac{\epsilon}{a}.$$

For $\tilde{R}$, we can easily find $r \in \mathbb{R}^{p(\tilde{N}-N)}$ with $|r|_\infty = a$ such that $|\tilde{R} r|_\infty = |\tilde{R}|_\infty \cdot a$. In fact, we can in addition pick r in such a way that

$$\left|\sum_{i=0}^{N} R_{\tilde{N},i}\xi(i) + \tilde{R} r\right|_\infty \ge |\tilde{R} r|_\infty = |\tilde{R}|_\infty \cdot a.$$

With r constructed as above, we can use it to define $\tilde{\xi}$ as follows:

$$\tilde{\xi}(k) := \xi(k) \qquad \text{for } k = 0, \cdots, N$$
$$\left[\tilde{\xi}(N+1)^T \cdots \tilde{\xi}(\tilde{N})^T\right]^T := r$$
$$\tilde{\xi}(k) := 0 \qquad k > \tilde{N}.$$

From this definition it is clear that $P_N \tilde{\xi} = \xi$ and that $|\tilde{\xi}|_\infty = a$. Finally

$$\begin{aligned} |(R\tilde{\xi})(\tilde{N})|_\infty &= \left|\sum_{i=0}^{\tilde{N}} R_{\tilde{N},i}\tilde{\xi}(i)\right|_\infty \\ &= \left|\sum_{i=0}^{N} R_{\tilde{N},i}\xi(i) + \sum_{i=N+1}^{\tilde{N}} R_{\tilde{N},i}\tilde{\xi}(i)\right|_\infty \\ &= \left|\sum_{i=0}^{N} R_{\tilde{N},i}\xi(i) + \tilde{R} r\right|_\infty \ge |\tilde{R}|_\infty \cdot a \\ &\ge \|S_{-(N+1)} R S_{(N+1)}\| \cdot a - \epsilon. \end{aligned}$$

This implies that

$$\|P_{\tilde{N}} R \tilde{\xi}\|_\infty \ge \|S_{-(N+1)} R S_{(N+1)}\| \cdot a - \epsilon.$$

■

Lemma 3: Let $R \in \mathscr{L}_{TV}^{q \times p}$. Then $(I - R\Delta)^{-1}$ is l^∞-stable for all $\Delta \in \Delta(p, q)$ if and only if there exists an integer N such that $\|S_{-N} R S_N\| < 1$.

Proof: We will prove the lemma by showing that $(I - R\Delta)^{-1}$ is *not* l^∞-stable for some $\Delta \in \Delta(p, q)$ if and only if $\|S_{-n} R S_n\| \ge 1$ for all $n \ge 0$. Using Lemma 1, the task of proving this lemma reduces to that of showing

$$\text{there exists } \xi \in l_{p,e}^\infty \setminus l_p^\infty \text{ and } c > 0 \text{ such that}$$
$$\|P_k \xi\|_\infty \le \|P_{k-1} R \xi\|_\infty + c \qquad \forall k \ge 0$$
$$\Updownarrow$$
$$\|S_{-n} R S_n\| \ge 1 \qquad \text{for all } n \ge 0. \qquad (5.1)$$

($\Rightarrow$) Assume (5.1) holds. It follows that for any fixed integer $n \ge 0$

$$\begin{aligned} \|P_k \xi\|_\infty &\le \|P_{k-1} R(S_n S_{-n} \xi + P_{n-1}\xi)\|_\infty + c \\ &\le \|P_{k-1} R S_n S_{-n} \xi\|_\infty + c + c' \qquad \text{for some } c' > 0. \end{aligned}$$

Using this together with

$$\|P_k S_{-n} \xi\|_\infty \le \|P_{k+n} \xi\|_\infty \text{ and}$$
$$\|P_{k+n-1} R S_n S_{-n} \xi\|_\infty \le \|P_{k-1} S_{-n} R S_n S_{-n} \xi\|_\infty + c'' \qquad \text{for some } c'' > 0 \text{ and } \forall k \ge 0$$

we get

$$\|P_k S_{-n} \xi\|_\infty \le \|P_{k-1} S_{-n} R S_n S_{-n} \xi\|_\infty + c + c' + c'' \qquad \text{for some } c'' > 0 \text{ and } \forall k \ge 0.$$

Define $\tilde{c} := c + c' + c''$. We can now write

$$\begin{aligned} \|P_k S_{-n} \xi\|_\infty &\le \|P_{k-1} S_{-n} R S_n S_{-n} \xi\|_\infty + \tilde{c} \\ &\le \|S_{-n} R S_n\| \, \|P_{k-1} S_{-n} \xi\|_\infty + \tilde{c} \\ &\le \|S_{-n} R S_n\| \, \|P_k S_{-n} \xi\|_\infty + \tilde{c}. \end{aligned}$$

Hence

$$\|P_k S_{-n} \xi\|_\infty (1 - \|S_{-n} R S_n\|) \le \tilde{c} \qquad \forall k \ge 0$$

which, since $\lim_{k \to \infty} \|P_k S_{-n} \xi\|_\infty = \infty$, implies that $\|S_{-n} R S_n\| \ge 1$. Since n was arbitrary, it follows that $\|S_{-n} R S_n\| \ge 1$ for all $n \ge 0$.

($\Leftarrow$) To prove the other direction, we assume $\|S_{-n} R S_n\| \ge 1$ for all n and then show the existence of $\xi \in l_{p,e}^\infty \setminus l_p^\infty$ and $c > 0$ such that ξ satisfies (5.1). This is done by first constructing a sequence of truncated elements of l_p^∞, namely $\{\xi_i\}_{i=1}^\infty$, and then defining ξ in terms of this sequence and verifying it has the desired properties. The construction of $\{\xi_i\}_{i=1}^\infty$ goes as follows: fix ϵ to be any real number greater than zero. Next, let $\xi_0 := 0 \in P_{-1}(l_p^\infty)$, and apply Lemma 2 to ξ_o with $a = 1$, to obtain an integer $N_1 > 0$ and $\xi_1 \in P_N(l_p^\infty)$. To this new sequence apply Lemma 2 again, this time with $a = 2$, to obtain an integer $N_2 > N_1$ and $\xi_2 \in P_{N_2}(l_p^\infty)$. Repeating this procedure indefinitely gives the sequence $\{\xi_i\}_{i=1}^\infty$ whose elements satisfy

1) $\xi_i \in P_{N_i}(l_p^\infty)$ for some integer $N_i > N_{i-1} > 0$,
2) $P_{N_i}\xi_i = \xi_{i-1} \quad i \ge 2$,
3) $\|\xi_i\|_\infty = i$,
4) $\|P_{N_{i-1}} R \xi_i\| \ge \|S_{-(N_{i-1}+1)} R S_{(N_{i-1}+1)}\| \cdot i - \epsilon \ge i - \epsilon$.

Next, we show by induction that for $i \ge 1$

$$\|P_k \xi_i\|_\infty \le \|P_{k-1} R \xi_i\|_\infty + c \qquad \text{for some } c > 0 \quad (5.2)$$

where c does not depend on i. Hence, let $c := 1 + \epsilon$. For $i = 1$, (5.2) holds trivially since

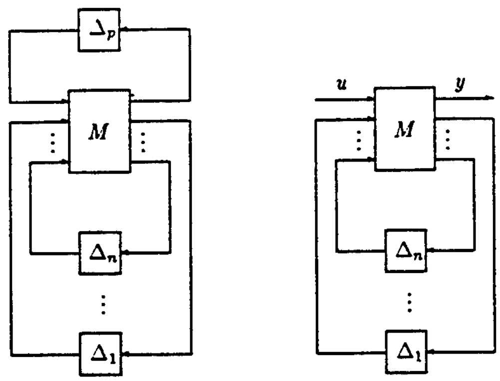
$$\|P_k \xi_1\|_\infty \le 1 \le \|P_{k-1} R \xi_1\|_\infty + c.$$

Next, assume (5.2) holds for $i = m$. We now show it must hold for $i = m + 1$. If $k \le N_m$ we have

$$\begin{aligned} \|P_k \xi_{m+1}\|_\infty = \|P_k \xi_m\|_\infty &\le \|P_{k-1} R \xi_m\|_\infty + c \\ &\le \|P_{k-1} R \xi_{m+1}\|_\infty + c. \end{aligned}$$

If, however, $k > N_m$, we can write

$$\begin{aligned} \|P_k \xi_{m+1}\|_\infty \le m + 1 &\le \|P_{N_m} R \xi_m\|_\infty + \epsilon + 1 \\ &\le \|P_{k-1} R \xi_{m+1}\|_\infty + c \end{aligned}$$



SYSTEM I SYSTEM II

Fig. 3. Equivalence of stability and performance robustness.

which completes the induction proof.

Finally, we define the desired $\xi \in l^{\infty}_{p,e} \setminus l^{\infty}_{p}$ to be the componentwise limit of such sequence, i.e.,

$$\xi(k) := \lim_{i \to \infty} \xi_i(k) \qquad \forall k \geq 0$$

which exists because of property (2). It is easy to see that ξ as defined here is the one we are looking for. Specifically, ξ belongs to $l^{\infty}_{p,e} \setminus l^{\infty}_{p}$ since $\lim_{i\to\infty} \|\xi_i\|_\infty = \infty$. In addition, ξ satisfies

$$\|P_k \xi\|_\infty \leq \|P_{k-1} R \xi\|_\infty + c \qquad \forall k \geq 0$$

which is inherited from the ξ_i's. This completes the proof. ■

We are now ready to state a theorem establishing a relation between stability robustness and performance robustness. It states that performance robustness in one system is equivalent to stability robustness in another one formed by adding a fictitious perturbation. A similar result has been shown to hold in [7] when the perturbations are linear time invariant and when the 2-norm is used to characterize the perturbation class. The same proof does not apply here though, due to the assumed time-varying nature of the perturbations. The usefulness of this theorem stems from the fact that we can now concentrate on finding conditions for achieving stability robustness alone. Once we do, performance robustness comes for free.

Consider the two systems shown in Fig. 3, where $M \in \mathscr{L}_{TI}^{q \times p}$ and $\Delta_i \in \Delta(p_i, q_i)$. In System II, u is a vector input of size $\tilde{p}$ and y is an output vector of size $\tilde{q}$. In System I, $\Delta_p \in \Delta(\tilde{p}, \tilde{q})$. It follows that $p = \tilde{p} + \sum_i p_i$ and $q = \tilde{q} + \sum_i q_i$. Subdivide M in the following manner:

$$M = \begin{pmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{pmatrix}$$

where $\tilde{M}_{11} \in \mathscr{L}_{TI}^{\tilde{q} \times \tilde{p}}$.

We now state the following theorem establishing the relation between System I and System II.

Theorem 1: The following four statements are equivalent.

1) System I achieves robust stability.

2) $(I - M\tilde{D})^{-1}$ is l^∞-stable for all $\tilde{D} \in \mathscr{D}[(\tilde{p}, \tilde{q}); (p_1, q_1); \cdots; (p_n, q_n)]$.

3) $(I - \tilde{M}_{22} D)^{-1}$ is l^∞-stable and $\|\tilde{M}_{11} + \tilde{M}_{12} D(I - \tilde{M}_{22} D)^{-1} \tilde{M}_{21}\| < 1$, for all D belonging to $\mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$.

4) System II achieves robust performance.

Proof: 1) ⇔ 2) follows from the remarks preceding Proposition 1. 3) ⇔ 4) is immediate since a necessary and sufficient condition for System II to be robustly stable is that $(I - \tilde{M}_{22} D)^{-1}$ is l^∞-stable for all D in $\mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$. Robust performance means that $\|\tilde{M}_{11} + \tilde{M}_{12} D(I - \tilde{M}_{22} D)^{-1} \tilde{M}_{21}\| < 1$ for all D in $\mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$, which is exactly what the remaining part of 3) states.

To prove the theorem, we therefore have to show that 2) ⇔ 3). Before we do that, we introduce the following notation:

$$M_{yu}(D) := \tilde{M}_{11} + \tilde{M}_{12} D(I - \tilde{M}_{22} D)^{-1} \tilde{M}_{21}.$$

We start by showing 3) ⇒ 2). So let $\tilde{D} \in \mathscr{D}[(\tilde{p}, \tilde{q}); (p_1, q_1); \cdots; (p_n, q_n)]$. $\tilde{D}$ can be written as diag (Δ_p, D) where $\Delta_p \in \Delta(\tilde{p}, \tilde{q})$ and $D \in \mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$. It can be easily checked that

$$(I - M\tilde{D})^{-1} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

where

$$\begin{aligned} N_{11} &:= (I - M_{yu}(D)\Delta_p)^{-1} \\ N_{12} &:= (I - M_{yu}(D)\Delta_p)^{-1} \tilde{M}_{12} D(I - \tilde{M}_{22} D)^{-1} \\ N_{21} &:= (I - \tilde{M}_{22} D)^{-1} \tilde{M}_{21} \Delta_p (I - M_{yu}(D)\Delta_p)^{-1} \\ N_{22} &:= (I + N_{21} \tilde{M}_{12} D)(I - \tilde{M}_{22} D)^{-1}. \end{aligned}$$

Since $\|M_{yu}(D)\| < 1$, it follows by the small gain theorem that $(I - M_{yu}(D)\Delta_p)^{-1}$ is l^∞-stable, which, in turn, implies that $(I - M\tilde{D})^{-1}$ itself is stable.

Before proving 2) ⇒ 3), we will first show that given any $D \in \mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$ we can find $\bar{D} \in \mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$ such that

$$\|S_{-n} M_{yu}(\bar{D}) S_n\| \geq \|M_{yu}(D)\| \qquad \forall n \geq 0.$$

To do this, we construct $\bar{D}$ explicitly. So let D be represented by its pulse response matrix, i.e.,

$$D = \begin{pmatrix} D_{00} & 0 & \cdots & \cdots \\ D_{10} & D_{11} & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Then $\bar{D}$ will be defined in terms of its pulse response matrix as follows:

$$\bar{D} := \begin{pmatrix} D_{00} & & & & & & \\ & D_{00} & & & 0 & & \\ & D_{10} & D_{11} & & & & \\ & & & D_{00} & & & \\ & & 0 & D_{10} & D_{11} & & \\ & & & D_{20} & D_{21} & D_{22} & \\ & & & & & & \ddots \end{pmatrix}.$$

It may be verified that the structure of the previous matrix ensures that $\bar{D} \in \mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$. Furthermore, it is

not difficult to see that

$$P_i S_{-m_i} \bar{D} S_{m_i} = P_i D \qquad \forall i \geq 0$$

where $m_i = m_{i-1} + i$ and $m_{-1} = 0$.

We now show that

$$\| S_{-n} M_{yu}(\bar{D}) S_n \| \geq \| M_{yu}(D) \| \qquad \forall n \geq 0.$$

It will suffice to prove that

$$\| S_{-n} M_{yu}(\bar{D}) S_n \| \geq \| P_k M_{yu}(D) \| \qquad \forall n, k \geq 0.$$

Hence, given $n \geq 0$ and $k \geq 0$, i can be chosen large enough such that $m_i > n$ and $i > k$. We can now write

$$\begin{aligned} \| S_{-n} M_{yu}(\bar{D}) S_n \| &\geq \| S_{-m_i} M_{yu}(\bar{D}) S_{m_i} \| \\ &\geq \| P_i S_{-m_i} M_{yu}(\bar{D}) S_{m_i} \| \\ &= \| P_i M_{yu}(P_i S_{-m_i} \bar{D} S_{m_i}) \| \\ &= \| P_i M_{yu}(P_i D) \| \\ &= \| P_i M_{yu}(D) \| \\ &\geq \| P_k M_{yu}(D) \| \end{aligned}$$

where we have used the time invariance of $\tilde{M}_{ij}$. This proves our claim. We can now use this fact to finish the proof of the theorem which we do by contradiction. Suppose that $(I - M\tilde{D})^{-1}$ is l^∞-stable for all $\tilde{D} \in \mathscr{D}[(\tilde{p}, \tilde{q}); (p_1, q_1); \cdots; (p_n, q_n)]$, but that for some $D_o \in \mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$ it holds that $\| M_{yu}(D_o) \| \geq 1$. We can then form $\bar{D}_o$ in $\mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$ as shown before which satisfies

$$\| S_{-n} M_{yu}(\bar{D}_0) S_n \| \geq \| M_{yu}(D_o) \| \geq 1 \qquad \forall n \geq 1.$$

By Lemma 3, this says that $(I - M_{yu}(\bar{D}_o)\Delta_p)^{-1}$ is not l^∞-stable for some Δ_p in $\Delta(\tilde{p}, \tilde{q})$ which contradicts the fact that $(I - M\tilde{D})^{-1}$ is l^∞-stable for all $\tilde{D} \in \mathscr{D}[(\tilde{p}, \tilde{q}); (p_1, q_1); \cdots; (p_n, q_n)]$. This completes the proof. ■

VI. Conditions for Stability Robustness

It has been shown in Section V that we can convert a performance robustness problem into one which involves stability robustness alone. We can therefore concentrate only on stability robustness. We seek nonconservative conditions for achieving stability robustness which are easy to verify. Before we begin, we establish some notational conventions. Throughout this section, the perturbation set will be $\mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$ for some positive integers $p_1, \cdots, p_n$ and $q_1, \cdots, q_n$. M belongs to $\mathscr{L}_{TI}^{q \times p}$ where $p := \sum_i p_i$ and $q := \sum_i q_i$. Hence, M can be partitioned as follows:

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix}$$

where M_{ij} has size $q_i \times p_i$. We also define two maps E_i and $\hat{E}_i$, which will depend on the p_i's and the q_i's, as follows:

$$E_i: l^\infty_{p,e} \to l^\infty_{pi,e} \text{ such that } E_i(\eta_1, \cdots, \eta_n) = \eta_i$$
$$\text{where } \eta_k \in l^\infty_{pk,e} \quad \text{for } k = 1, \cdots, n$$

$$\hat{E}_i: l^\infty_{q,e} \to l^\infty_{qi,e} \text{ such that } \hat{E}_i(\hat{\eta}_1, \cdots, \hat{\eta}_n) = \hat{\eta}_i$$
$$\text{where } \hat{\eta}_k \in l^\infty_{qk,e} \quad \text{for } k = 1, \cdots, n.$$

The next lemma is crucial in the proof of the theorem to follow. It states necessary and sufficient conditions for a sequence in l_q^∞ to be mapped to another in l_p^∞ by a linear strictly causal map with norm less than or equal to one.

Lemma 4: Let $x = \{x(i)\}_{i=0}^\infty \in l^\infty_{m,e}$ and $y = \{y(i)\}_{i=0}^\infty \in l^\infty_{r,e}$. There exists $\Delta \in \Delta(r, m)$ such that $\Delta y = x$ if and only if

$$\| P_k x \|_\infty \leq \| P_{k-1} y \|_\infty \qquad \forall k \geq 0. \tag{6.1}$$

Proof: The necessity is immediate, so we proceed to the sufficiency part. So assume that (6.1) holds. The proof is trivial if $y = 0$: just pick Δ itself to be zero. So assume $y \neq 0$. We will now *construct* a Δ that has the desired properties. We start by identifying a subset of the $y(i)$'s, call it $y(i_1), y(i_2), \cdots$, which, depending on y, may or may not be finite. This subset may be defined recursively in the following manner: Let i_1 be the smallest integer such that $y(i_1) \neq 0$. Given $y(i_n)$, let i_{n+1} be the smallest integer greater that i_n such that $| y(i_{n+1}) |_\infty \geq | y(i_n) |_\infty$. Using the $x(i)$'s and $y(i_j)$'s we are now ready to construct Δ through specifying its pulse response matrix. So define

$$\Delta := \begin{pmatrix} \Delta_{00} & \Delta_{01} & \Delta_{02} & \cdots \\ \Delta_{10} & \Delta_{11} & \Delta_{12} & \cdots \\ \Delta_{20} & \Delta_{21} & \Delta_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$\Delta_{i_1+1, i_1} := \left[\frac{x(i_1+1)}{y(i_1)} \right], \Delta_{i_1+2, i_1} := \left[\frac{x(i_1+2)}{y(i_1)} \right],$$
$$\cdots, \Delta_{i_2, i_1} := \left[\frac{x(i_2)}{y(i_1)} \right],$$
$$\Delta_{i_2+1, i_2} := \left[\frac{x(i_2+1)}{y(i_2)} \right], \Delta_{i_2+2, i_2} := \left[\frac{x(i_2+2)}{y(i_2)} \right],$$
$$\cdots, \Delta_{i_3, i_2} := \left[\frac{x(i_3)}{y(i_2)} \right],$$
$$\Delta_{i_3+1, i_3} := \left[\frac{x(i_3+1)}{y(i_3)} \right], \cdots$$

and 0 otherwise.

Notice that each row of any of these matrices has at most one nonzero element, which, by the choice of the $y(i_j)$'s, will have its absolute value less than or equal to one.

Δ will have the form

$$\Delta = \begin{bmatrix} \ddots & & & & & & & & \\ \vdots & 0 & & & & & 0 & & \\ \vdots & \Delta_{i_1+1,i_1} & 0 & & & & & & \\ & \vdots & \vdots & \ddots & & & & & \\ & \Delta_{i_2,i_1} & 0 & \cdots & 0 & & & & \\ & 0 & & \vdots & \Delta_{i_2+1,i_2} & 0 & & & \\ & & & & \vdots & \vdots & \ddots & & \\ & & 0 & & \Delta_{i_3,i_2} & 0 & \cdots & 0 & \\ & & & & 0 & \vdots & & \Delta_{i_3+1,i_3} & \\ & & & & \vdots & \vdots & & \vdots & \ddots \end{bmatrix}$$

from which it is easy to see that Δ is causal and that $\Delta x = y$. Furthermore, it follows from the remarks immediately following the definition of Δ that

$$\|\Delta\| = \sup_i |(\Delta_{i,1} \quad \Delta_{i,2} \quad \cdots)|_\infty \le 1.$$

This completes the proof. ∎

Using this lemma, we now state an alternative condition for $(I - MD)^{-1}$ to be l^∞-stable for all D in $\mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$.

Theorem 2: There exists $D \in \mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$ such that $(I - MD)^{-1}$ is not l^∞-stable if and only if there exists $\xi \in l^\infty_{p,e} \setminus l^\infty_p$ and $c > 0$ such that

$$\|P_k E_i \xi\|_\infty - \|P_{k-1}\hat{E}_i(M\xi)\|_\infty \le c \qquad \text{for } i = 1, \ldots, n \text{ and } \forall k \ge 0.$$

Proof:

$(I - MD)^{-1}$ is not l^∞-stable

for some $D \in \mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$

$$\Updownarrow$$

$(I - MD)$ is not invertible as a map from l^∞_q to l^∞_q

for some $D \in \mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$

$$\Updownarrow$$

$\exists y \in l^\infty_{q,e} \setminus l^\infty_q$ such that $(I - MD)y \in l^\infty_q$

for some $D \in \mathscr{D}[(p_1, q_1); \cdots; (p_n, q_n)]$

$$\Updownarrow$$

$\exists \xi \in l^\infty_{p,e} \setminus l^\infty_p$ and $y \in l^\infty_{q,e} \setminus l^\infty_q$ such that

$$y - M\xi \in l^\infty_q \text{ and } \|P_k E_i \xi\|_\infty \le \|P_{k-1}\hat{E}_i y\|_\infty \qquad (6.2)$$

where we have made use of the open mapping theorem to conclude the second statement from the first, and Lemma 4 to get the last statement above. To finish the proof, we will show that (6.2) is equivalent to the following:

$\exists \xi \in l^\infty_{p,e} \setminus l^\infty_p$ and $c > 0$ such that

$$\|P_k E_i \xi\|_\infty - \|P_{k-1}\hat{E}_i M\xi\|_\infty \le c \qquad \forall k \ge 0 \text{ and } i = 1, \cdots, q. \qquad (6.3)$$

Therefore, assume (6.2) holds. It follows that there exists $c > 0$ such that

$$c \ge \|(y - M\xi)\|_\infty.$$

For such ξ and c, we can now write

$$\begin{aligned} c &\ge \|P_{k-1}\hat{E}_i(y - M\xi)\|_\infty \\ &\ge \|P_{k-1}\hat{E}_i y\|_\infty - \|P_{k-1}\hat{E}_i(M\xi)\|_\infty \\ &\ge \|P_k E_i \xi\|_\infty - \|P_{k-1}\hat{E}_i(M\xi)\|_\infty \\ &\quad \forall k \ge 0 \text{ and } i = 1, \ldots, q \end{aligned}$$

which is exactly what (6.3) states. Conversely, assume (6.3) holds. Define

$$(\pi_i y)(k) := (\pi_i M\xi)(k) + c\,\mathrm{sgn}(\pi_i M\xi)(k) \qquad \forall k \text{ and } i = 1, \cdots, q.$$

It follows from this definition that $y - M\xi$ is in l^∞_q. Furthermore, it is immediate from the definition of y that

$$\|P_{k-1}\hat{E}_i y\|_\infty = \|P_{k-1}\hat{E}_i(M\xi)\|_\infty + c$$

which, together with (6.3), gives

$$\|P_k E_i \xi\|_\infty \le \|P_{k-1}\hat{E}_i y\|_\infty.$$

From this last equation, we also get that $y \in l^\infty_{q,e} \setminus l^\infty_q$. This proves the theorem. ∎

Proof of Lemma 1: The proof of Lemma 1, as may be readily checked, is identical to that of Theorem 2 with M replaced by R and with $n = 1$. Note that even though M has been assumed to be time invariant, this was never used in the proof of Theorem 2. ∎

Before we can state Theorem 3, we need to establish two additional lemmas. Together with Theorem 2, these lemmas will be the main tools used in the proof Theorem 3.

Lemma 5: Let $\xi \in P_N(l^\infty_p) = P_N(l^\infty_{p_1}) \times \cdots \times P_N(l^\infty_{p_n})$. Then given $\epsilon > 0$ and $a \in \mathbb{R}^n$ such that $a_i \ge \|E_i \xi\|_\infty$, there exists an integer $\tilde{N} > N$ and $\tilde{\xi} \in P_{\tilde{N}}(l^\infty_p)$ such that

$$\|E_i \tilde{\xi}\|_\infty = a_i$$

$$P_N E_i \tilde{\xi} = E_i \xi \qquad \text{and}$$

$$\|P_{\tilde{N}}\hat{E}_i M\tilde{\xi}\|_\infty \ge \max_{1 \le m \le q_i} \left\{ \sum_{j=1}^{n} \|\pi_m M_{ij}\|_{\mathscr{A}} a_j \right\} - \epsilon$$

$$i = 1, \cdots, n.$$

Proof: This lemma follows immediately from [5, Lemma 5.2]. ■

Lemma 6: Let c and a_{ij}, where $i = 1, \cdots, n$ and $j = 1, \cdots, n$, be nonnegative real numbers. Then the following is true.

There exist n sequences of nonnegative real numbers, say $\eta_i = \{\eta_i(k)\}_{k=0}^{\infty}$, at least one of which is unbounded, satisfying

$$\eta_i(k) \leq \sum_{j=1}^{n} a_{ij}\eta_j(k) + c \qquad \forall k \text{ and } i = 1, \cdots, n \tag{6.4}$$

$$\Updownarrow$$

the system of inequalities

$$x_i \leq \sum_{j=1}^{n} a_{ij}x_j \qquad i = 1, \cdots, n \tag{6.5}$$

has a solution in $(\mathbb{R}^+)^n \setminus \{0\}$.

Proof: It is immediate that (6.5) ⇒ (6.4); merely let $\eta_i(k) = k\bar{x}_i$ where $(\bar{x}_1, \cdots, \bar{x}_n)$ is the solution guaranteed by (6.5).

We prove the other direction by induction on n. For $n = 1$, (6.4) gives an unbounded sequence of real numbers η such that

$$\eta(k) \leq a\eta(k) + c \qquad \forall k.$$

Since $\limsup_k \eta(k) = \infty$, it follows that $a \geq 1$. This, in turn, is true if and only if $x \leq ax$ has a solution in $(0, \infty)$, or equivalently if and only if (6.5) holds for $n = 1$.

For the second induction step, assume the lemma is true whenever $n \leq n_o - 1$. Assuming (6.4) holds for real numbers c and a_{ij} where $i = 1, \cdots, n_o$ and $j = 1, \cdots, n_o$ we will complete the proof by showing that (6.5) must hold as well. We start by noting that without loss of generality the unbounded sequence guaranteed by assuming (6.4) has index less than n_o, i.e., for some $i < n_o$, we have $\limsup_{k\to\infty}\eta_i(k) = \infty$. If this were not the case, we can always renumber the η_i's. We may also assume that $a_{n_on_o} < 1$, otherwise the proof of this lemma is complete since $\bar{x} = (0, \cdots, 1)$ solves the system of inequalities in (6.5) for $n = n_o$. Based on this last assumption, dividing both sides of the n_oth inequality of (6.4) by $1 - a_{n_on_o}$ yields

$$\eta_{n_o}(k) \leq \frac{1}{1 - a_{n_on_o}} \sum_{j=1}^{n_o - 1} a_{n_oj}\eta_j(k) + \frac{c}{1 - a_{n_on_o}} \qquad \forall k.$$

Substituting this inequality in the first $n_o - 1$ inequalities of (6.4), we get

$$\eta_i(k) \leq \sum_{j=1}^{n_o-1}\left(a_{ij} + \frac{a_{in_o}a_{n_oj}}{1 - a_{n_on_o}}\right)\eta_j(k) + c' \qquad \forall k \text{ and } i = 1, \cdots, n_o - 1$$

where $c' = c + c/(1 - a_{n_on_o}) \geq 0$. But by the induction hypothesis this implies that the system

$$x_i \leq \sum_{j=1}^{n_o-1}\left(a_{ij} + \frac{a_{in_o}a_{n_oj}}{1 - a_{n_on_o}}\right)x_j \qquad i = 1, \cdots, n_o - 1$$

has a solution in $(\mathbb{R}^+)^{n_o - 1} \setminus \{0\}$. Denote this solution by $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_{n_o-1})$. Now define

$$\bar{x}_{n_o} := \sum_{j=1}^{n_o-1} \frac{a_{n_oj}}{1 - a_{n_on_o}}\bar{x}_j \in \mathbb{R}^+.$$

Clearly, $(\bar{x}_1, \cdots, \bar{x}_{n_o}) \in (\mathbb{R}^+)^{n_o} \setminus \{0\}$, and it can be easily checked that it solves the inequalities in (6.5) for $n = n_o$. This completes the induction proof. ■

Next, we will state our main result establishing the equivalence of the stability robustness problem to a simple algebraic one. Depending on the region in which this algebraic problem has its solutions, we can conclude whether or not our system achieves robust stability, and by the results of the previous section, robust performance. In order not to clutter the exposition, we first state and prove this theorem in the scalar case. Hence $p_1 = \cdots = p_n = q_1 = \cdots = q_n = 1$. Consequently, for any i, E_i will be equal to π_i.

Theorem 3: $(I - MD)^{-1}$ is not l^∞-stable for some $D \in \mathscr{D}[(1, 1); \cdots; (1, 1)]$ if and only if the system

$$x_i \leq \sum_{j=1}^{n} \|M_{ij}\|_{\mathscr{A}} x_j \qquad i = 1, \cdots, n$$

has a solution $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)$ in $(\mathbb{R}^+)^n \setminus \{0\}$.

Proof: Assume $(I - MD)^{-1}$ is not l^∞-stable for some $D \in \mathscr{D}[(1, 1); \cdots; (1, 1)]$. By Theorem 2, there exists $c > 0$ and $\xi \in l^\infty_{n,e} \setminus l^\infty_n$ such that

$$\|P_kE_i\xi\|_\infty - \|P_{k-1}\sum_{j=1}^{n} M_{ij}E_j\xi\|_\infty \leq c \qquad \forall k \geq 0, \text{ and } i = 1, \cdots, n.$$

Applying the triangle inequality and using the causality of M_{ij}, and the fact that the projection operator is contractive we get

$$\|P_kE_i\xi\|_\infty \leq \sum_{j=1}^{n} \|M_{ij}\|_{\mathscr{A}} \|P_kE_j\xi\|_\infty + c \qquad i = 1, \cdots, n.$$

Finally, applying Lemma 6 gives the desired conclusion.

For the other direction, assume there exists $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n) \in (\mathbb{R}^+)^n \setminus \{0\}$ such that

$$\bar{x}_i \leq \sum_{j=1}^{n} \|M_{ij}\|_{\mathscr{A}} \bar{x}_j \qquad i = 1, \cdots, n.$$

We will show that this will imply the existence of $c > 0$ and $\xi \in l^\infty_{n,e} \setminus l^\infty_n$ such that

$$\|P_kE_i\xi\|_\infty - \|P_{k-1}\hat{E}_iM\xi\|_\infty < c \qquad i = 1, \cdots, n$$

which by Theorem 2 implies that $(I - MD)^{-1}$ is not l^∞-stable for some $D \in \mathscr{D}[(1, 1); \cdots; (1, 1)]$. We start by defining a sequence of elements of l^∞_p as follows: let $\xi_o := 0 \in l^\infty_p$. Fix $\epsilon > 0$ to be any real number, and define $c := \max_i \bar{x}_i + \epsilon$. Now apply Lemma 5 to ξ_o with $a = (\bar{x}_1, \cdots, \bar{x}_n)$ to get $N_1 > 0$ and $\xi_1 \in P_{N_1}(l^\infty_p)$. Next, apply the same lemma again to ξ_1 with $a = (2\bar{x}_1, \cdots, 2\bar{x}_n)$ to get $N_2 > N_1$ and $\xi_2 \in P_{N_2}(l^\infty_p)$. Repeating this procedure indefinitely we get the sequence $\{\xi_m\}_{m=0}^\infty$ with the properties

$$\xi_m \in P_{N_m}(l^\infty_p) \qquad \text{where } N_m > N_{m-1} > \cdots$$

$$P_{N_{m-1}}\xi_m = \xi_{m-1}$$

$$\|E_i\xi_m\|_\infty = m\bar{x}_i, \qquad \text{and}$$

$$\|P_{N_m}E_iM\xi_m\|_\infty \geq \sum_{j=1}^{n} \|M_{ij}\|_{\mathscr{A}} m\bar{x}_j - \epsilon \qquad \forall k \geq 0 \text{ and for } i = 1, \cdots, n.$$

TABLE I
CONDITIONS FOR STABILITY/PERFORMANCE ROBUSTNESS FOR $n = 1, 2,$ AND 3

No. of Δ's	Necessary and Sufficient Conditions for Stability Robustness
$n = 1$	$\Vert M\Vert_{\mathcal{A}} < 1$
$n = 2$	$\Vert M_{22}\Vert_{\mathcal{A}} < 1$ $\Vert M_{11}\Vert_{\mathcal{A}} + \dfrac{\Vert M_{12}\Vert_{\mathcal{A}}\Vert M_{21}\Vert_{\mathcal{A}}}{1-\Vert M_{22}\Vert_{\mathcal{A}}} < 1$
$n = 3$	$\Vert M_{33}\Vert_{\mathcal{A}} < 1$ $\Vert M_{22}\Vert_{\mathcal{A}} + \dfrac{\Vert M_{23}\Vert_{\mathcal{A}}\Vert M_{32}\Vert_{\mathcal{A}}}{1-\Vert M_{33}\Vert_{\mathcal{A}}} < 1$ $\Vert M_{11}\Vert_{\mathcal{A}} + \dfrac{\Vert M_{13}\Vert_{\mathcal{A}}\Vert M_{31}\Vert_{\mathcal{A}}}{1-\Vert M_{33}\Vert_{\mathcal{A}}} + \dfrac{\left(\Vert M_{12}\Vert_{\mathcal{A}} + \dfrac{\Vert M_{13}\Vert_{\mathcal{A}}\Vert M_{32}\Vert_{\mathcal{A}}}{1-\Vert M_{33}\Vert_{\mathcal{A}}}\right)\left(\Vert M_{21}\Vert_{\mathcal{A}} + \dfrac{\Vert M_{23}\Vert_{\mathcal{A}}\Vert M_{31}\Vert_{\mathcal{A}}}{1-\Vert M_{33}\Vert_{\mathcal{A}}}\right)}{1-\left(\Vert M_{22}\Vert_{\mathcal{A}} + \dfrac{\Vert M_{23}\Vert_{\mathcal{A}}\Vert M_{32}\Vert_{\mathcal{A}}}{1-\Vert M_{33}\Vert_{\mathcal{A}}}\right)} < 1$

We next show by induction that for any m

$$\|P_kE_i\xi_m\|_\infty \le \|P_{k-1}\hat{E}_iM\xi_m\|_\infty + c \qquad \forall k \ge 0 \text{ and for } i = 1,\cdots,n. \tag{6.6}$$

When $m = 0$, this is trivial since

$$\|P_kE_i\xi_o\|_\infty = 0 \le \|P_{k-1}\hat{E}_iM\xi_o\|_\infty + c \qquad \forall k \ge 0 \text{ and for } i = 1,\cdots,n.$$

For the second induction step, assume (6.6) holds for $m = m_o$. We now show that (6.6) holds for $m = m_o + 1$. We prove this in two parts: first when $k \le N_{m_o}$ and then when $k > N_{m_o}$.

For $k \le N_{m_o}$

$$\begin{aligned}\|P_kE_i\xi_{m_o+1}\|_\infty &= \|P_kE_i\xi_{m_o}\|_\infty \\ &\le \|P_{k-1}\hat{E}_iM\xi_{m_o}\|_\infty + c \\ &= \|P_{k-1}\hat{E}_iM\xi_{m_o+1}\|_\infty + c \qquad i = 1,\cdots,n.\end{aligned}$$

When $k > N_{m_o}$ we have

$$\begin{aligned}\|P_kE_i\xi_{m_o+1}\|_\infty &\le \|P_{Nm_o}E_i\xi_{m_o}\|_\infty + \max_j \bar{x}_j \\ &\le m_o\bar{x}_i + \max_j \bar{x}_j \\ &\le \sum_{j=1}^{n}\|M_{ij}\|_{\mathcal{A}}m_o\bar{x}_j + \max_j \bar{x}_j \\ &\le \|P_{Nm_o}\hat{E}_iM\xi_{m_o}\|_\infty + \epsilon + \max_j \bar{x}_j \\ &\le \|P_{k-1}\hat{E}_iM\xi_{m_o+1}\|_\infty + c \qquad i = 1,\cdots,n.\end{aligned}$$

This completes the induction proof, and thus we have

$$\|P_kE_i\xi_m\|_\infty \le \|P_{k-1}\hat{E}_iM\xi_m\|_\infty + c \qquad \forall k, m \ge 0 \text{ and for } i = 1,\cdots,n.$$

Finally, define ξ by letting $\xi(k) := \lim_{m\to\infty}\xi_m(k)$ for all k. It follows that

$$\|P_kE_i\xi\|_\infty \le \|P_{k-1}\hat{E}_iM\xi\|_\infty + c \qquad i = 1,\cdots,n.$$

Furthermore, $\xi \in l_{n,e}^\infty \setminus l_n^\infty$ since $\lim_{k\to\infty}\|P_k\xi\|_\infty = \infty$. This completes the proof. ∎

With this theorem, our problem stated in Section IV is essentially solved. Applying this theorem to the performance and stability robustness problem stated earlier, reduces it to a simple algebraic one in which the object is to determine whether a certain system of inequalities has a solution in a particular region in $\mathbb{R}^n$. What makes this algebraic problem particularly attractive is that the set of inequalities that arises relates in a simple and direct manner to the original problem. Only norms of the subentries of the M matrix arise and they do so in the same general order that they do in M. The question that arises, naturally at this point, is how can one determine whether the system of inequalities at hand has a solution in the related region of $\mathbb{R}^n$? It turns out, that no search techniques are needed to accomplish this task and the answer to this question can be determined by evaluating certain expressions directly. These expressions also involve norms of the subentries of M and thus are easy to compute. The derivation of these alternate conditions for stability and performance robustness is the next topic of discussion.

The first step in restating the conditions involving the set of inequalities is to make the following observation.

Observation: The system of inequalities

$$x_i \le \sum_{j=1}^{n}\|M_{ij}\|_{\mathcal{A}}x_j \qquad i = 1,\cdots,n$$

has a solution in $(\mathbb{R}^+)^n \setminus \{0\}$ if and only if either $\|M_{nn}\|_{\mathcal{A}} \ge 1$ or $\|M_{nn}\|_{\mathcal{A}} < 1$ and the system of inequalities

$$x_i \le \sum_{j=1}^{n-1}\left(\|M_{ij}\|_{\mathcal{A}} + \frac{\|M_{in}\|_{\mathcal{A}}\|M_{nj}\|_{\mathcal{A}}}{1-\|M_{nn}\|_{\mathcal{A}}}\right)x_j \qquad i = 1,\cdots,n-1$$

has a solution in $(\mathbb{R}^+)^{n-1} \setminus \{0\}$.

Notice that this observation allows us to replace the task of determining whether any solutions to a set of n inequalities lie in a certain region by the simpler one of determining whether the solutions to a set of $n-1$ inequalities lie in a small region together with a simple test on the norm of M_{nn}. It is easily seen how this can be repeated until we completely replace all such conditions by tests on expressions involving norms of the M_{ij}'s, a much simpler task. Table I lists some of these for a few values of n.

Until Theorem 3, all our derivations were done with an MIMO system in mind. Theorem 3 broke this trend in order to avoid the additional notational complexity which would undoubt-

edly obscure the ideas in the proof. We will now tie this loose end by stating without proof the analog of Theorem 3 in the MIMO case. All the tools for the proof have been developed and the steps are identical to those taken in the proof of Theorem 3.

In order to discuss the multivariable case we will need to make reference to the rows of M_{ij} which are themselves stable rational functions. Let us denote the mth row of M_{ij} by $(M_{ij})_m$, which is also equal to $\pi_m M_{ij}$. Since we will no longer restrict the p_i's and q_i's to be equal to one, the following set is not necessarily a singleton:

$$\mathscr{K} := \{(k_1,\cdots,k_n)\in Z^n \colon 1 \le k_i \le q_i\}.$$

From this definition it is clear that the set $\mathscr{K}$ has exactly $\Pi_{i=1}^n q_i$ elements. To each $k\in\mathscr{K}$ corresponds the system of inequalities: $x_i \le \sum_{j=1}^n \|(M_{ij})_{k_i}\|_{\mathscr{A}} x_j$ where $k = (k_1,\cdots,k_n)$. As the next theorem shows, it is the solutions of these inequalities that are of concern when seeking necessary and sufficient conditions for stability and performance robustness in the MIMO case.

Theorem 4: $(I - MD)^{-1}$ is not l^∞-stable for some $D \in \mathscr{D}[(p_1, q_1);\cdots;(p_n, q_n)]$ if and only if for some $k = (k_1,\cdots,k_n)\in\mathscr{K}$, the system

$$x_i \le \sum_{j=1}^n \|(M_{ij})_{k_i}\|_{\mathscr{A}} x_j, \qquad i = 1,\cdots,n$$

has a solution $\bar{x} = (\bar{x}_1,\cdots,\bar{x}_n)$ in $(\mathbb{R}^+)^n \setminus \{0\}$.

VII. Nonlinear Perturbations

In this section, it will be shown that if the class of perturbations is enlarged to include norm-bounded nonlinear perturbations, then the conditions for robust stability remain the same. This means that robustness to linear time-varying perturbations will automatically guarantee robustness to nonlinear perturbations as well. Furthermore, it is shown that when enlarging the perturbation class to include nonlinear perturbations, stability robustness remains equivalent to performance robustness, and so the conditions for stability and performance robustness for time-varying perturbations are the same as those for nonlinear perturbations. For simplicity, we shall consider the scalar case here. We start by extending our definition for the perturbation class to include nonlinear perturbation. So define

$$\mathscr{D}_{NL}[(p_1, q_1);\cdots;(p_n, q_n)] := \left\{ \operatorname{diag}(\Delta_1,\cdots,\Delta_n) \mid \Delta_i \text{ strictly causal and } \sup_{x\neq 0}\frac{\|\Delta_i x\|_\infty}{\|x\|_\infty} \le 1 \right\}.$$

For simplicity we adopt the following notation:

$$\mathscr{D}(n) := \mathscr{D}\Big[\overbrace{(1,1);\cdots;(1,1)}^{n}\Big]$$

$$\mathscr{D}_{NL}(n) := \mathscr{D}_{NL}\Big[\overbrace{(1,1);\cdots;(1,1)}^{n}\Big].$$

Theorem 5: $(I - MD)^{-1}$ is l^∞-stable for all $D\in\mathscr{D}(n)$ if and only if it is l^∞-stable for all $D \in \mathscr{D}_{NL}(n)$.

Proof: The sufficiency part is immediate since $\mathscr{D}(n) \subset \mathscr{D}_{NL}(n)$. We prove necessity by contradiction. Suppose $(I - MD)^{-1}$ is not l^∞-stable for some $D \in \mathscr{D}_{NL}(n)$. Then either $(I - MD)^{-1}$ does not map l_n^∞ into l_n^∞ or it does but $(I - MD)^{-1}$ is not bounded. Notice that the second possibility was eliminated when D was restricted to be in $\mathscr{D}(n)$ since in this case if $I - MD$ maps l_n^∞ onto l_n^∞ then $(I - MD)^{-1}$ is bounded by the open mapping theorem. Repeating the same arguments used before, the first of these possibilities can be shown equivalent to the system of inequalities

$$x_i \le \sum_{j=1}^n \|M_{ij}\|_{\mathscr{A}} x_j \qquad i = 1,\cdots,n$$

having a solution in $(\mathbb{R}^+)^n \setminus \{0\}$. By Theorem 3, this implies that $(I - MD)^{-1}$ is not l^∞-stable for some D in $\mathscr{D}(n)$ and hence in $\mathscr{D}_{NL}(n)$.

Now suppose $(I - MD)^{-1}$ maps l_n^∞ into l_n^∞ for all $D \in \mathscr{D}_{NL}(n)$ but that for some $D\in\mathscr{D}_{NL}(n)$ it is not bounded. This implies the existence of a sequence of elements of l_n^∞, $\{x_k\}_{k=0}^\infty$, with $x_k \neq 0$, such that

$$\lim_{k\to\infty}\frac{\|(I - MD)^{-1}x_k\|_\infty}{\|x_k\|_\infty} = \infty.$$

Define $y_k := (I - MD)^{-1}x_k$. From this definition we have

$$\pi_i y_k = \pi_i x_k + \sum_{j=1}^n M_{ij}\Delta_j \pi_j y_k.$$

Using the triangle inequality and dividing by $\|x_k\|_\infty$, we get

$$\eta_i(k) \le \sum_{j=1}^n \|M_{ij}\|_{\mathscr{A}} \eta_j(k) + 1$$

where $\eta_i(k) := \|\pi_i y_k\|_\infty / \|x_k\|_\infty$. Applying Lemma 6 gives us that the system of inequalities:

$$x_i \le \sum_{j=1}^n \|M_{ij}\|_{\mathscr{A}} x_j, \qquad i = 1,\cdots,n$$

has a solution in $(\mathbb{R}^+)^n \setminus \{0\}$. As before, Theorem 3 implies that $(I - MD)^{-1}$ is not l^∞-stable for some $D\in\mathscr{D}(n)$. ■

One last issue remains to be settled. We have shown that stability robustness is equivalent to performance robustness when the class of perturbations is $\mathscr{D}(n)$. It does not immediately follow that this should be true if the perturbation class were $\mathscr{D}_{NL}(n)$. Next, we shall show that indeed stability robustness is equivalent to performance robustness even when enlarging the perturbation class to include nonlinear perturbations.

We will assume the class of perturbations is $\mathscr{D}_{NL}(n)$ and that we have one performance objective consisting of keeping the norm of the function mapping the external input u to the output y less than one (Fig. 3, SYSTEM II).

Therefore, we have the following M matrix:

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1,n+1} \\ \vdots & & \vdots \\ M_{n+1,1} & \cdots & M_{n+1,n+1} \end{pmatrix}.$$

As before, we define the subentries of M as follows:

$$\tilde{M}_{11} := M_{11} \qquad \tilde{M}_{12} := (M_{12} \quad \cdots \quad M_{1,n+1})$$

$$\tilde{M}_{21} := \begin{pmatrix} M_{21} \\ \vdots \\ M_{n+1,1} \end{pmatrix}$$

$$\tilde{M}_{22} := \begin{pmatrix} M_{22} & \cdots & M_{2,n+1} \\ \vdots & & \vdots \\ M_{n+1,2} & \cdots & M_{n+1,n+1} \end{pmatrix}.$$

The next theorem establishes the equivalence of stability and performance robustness when nonlinear perturbations are included.

Theorem 6: $(I - \tilde{M}_{22}D)^{-1}$ is l^∞-stable and $\|\tilde{M}_{11} + \tilde{M}_{12}D(I - \tilde{M}_{22}D)^{-1}\tilde{M}_{21}\| < 1$ for all $D \in \mathcal{D}_{NL}(n)$ if and only if $(I - M\tilde{D})^{-1}$ is l^∞-stable for all $\tilde{D} \in \mathcal{D}_{NL}(n+1)$.

Proof:

(Only if): This direction is an immediate consequence of the small gain theorem.

(If): Clearly, if $(I - \tilde{M}_{22}D)^{-1}$ were not l^∞-stable for some $D \in \mathcal{D}_{NL}(n)$ then $(I - M\tilde{D})^{-1}$ will not be l^∞-stable for some $\tilde{D} \in \mathcal{D}_{NL}(n+1)$. So suppose $\|\tilde{M}_{11} + \tilde{M}_{12}D_o(I - \tilde{M}_{22}D_o)^{-1}\tilde{M}_{21}\| \ge 1$ for some $D_o \in \mathcal{D}_{NL}(n)$. Now define

$$L := \begin{pmatrix} L_2 \\ L_3 \\ \vdots \\ L_{n+1} \end{pmatrix} = D_o(I - \tilde{M}_{22}D_o)^{-1}\tilde{M}_{21}.$$

It follows that

$$1 \le \|\tilde{M}_{11} + \tilde{M}_{12}L\| \le \|M_{11}\|_{\mathcal{A}} + \|M_{12}\|_{\mathcal{A}}\|L_2\| + \cdots + \|M_{1,n+1}\|_{\mathcal{A}}\|L_{n+1}\|. \quad (7.1)$$

Using the triangle inequality and the submultiplicativity of the norm it follows from the definition of L that

$$\|L_2\| \le \|M_{21}\|_{\mathcal{A}} + \|M_{22}\|_{\mathcal{A}}\|L_2\| + \cdots + \|M_{2,n+1}\|_{\mathcal{A}}\|L_{n+1}\|$$

$$\vdots$$

$$\|L_{n+1}\| \le \|M_{n+1,1}\|_{\mathcal{A}} + \|M_{n+1,2}\|_{\mathcal{A}}\|L_2\| + \cdots + \|M_{n+1,n+1}\|_{\mathcal{A}}\|L_{n+1}\|.$$

Combining these inequalities with (7.1) it can be seen that $(1, \|L_2\|, \cdots, \|L_n + 1\|)$ solves the following system of inequalities:

$$x_i \le \sum_{j=1}^{n+1} \|M_{ij}\|_{\mathcal{A}} x_j \qquad i = 1, \cdots, n+1$$

which by Theorem 3 implies that $(I - M\tilde{D})^{-1}$ is not l^∞-stable for some $\tilde{D} \in \mathcal{D}_{NL}(n+1)$. This completes the proof. ■

VIII. Applications

In this section, we present some applications to the theory developed thus far. Starting with stability robustness, we provide necessary and sufficient conditions for stability robustness in the simple case when only one perturbation is considered. Next, we add a performance objective, namely the sensitivity function, and demonstrate how its norm can be made small in the presence of multiplicative plant perturbations, subject of course to robust stability. We contrast the conditions obtained when the *input* sensitivity function is the performance objective of interest, to those obtained when the *output* sensitivity is considered instead. Both of these cases involve two Δ's, one representing the actual plant perturbations, and the other fictitious, representing the performance objective to be achieved. Finally, we provide an example where three Δ's are involved. This example arises when one considers a class of plants formed by perturbing a nominal linear shift-invariant plant through adding both additive and multiplicative perturbations and demands that the worst case norm of the sensitivity function is to be minimized through the choice of a robustly stabilizing controller. We begin with the stability robustness application.

1) Stability Robustness (Unstructured): This is the simplest case. The perturbations take the form of one Δ having q inputs and p outputs. The question then is when is $(I - M\Delta)^{-1}$ stable for all Δ in $\Delta(p, q)$? Equivalently, when is the interconnection of $M \in \mathcal{L}_{TI}^{q \times p}$ and Δ stable for all Δ in $\Delta(p, q)$? From Theorem 4, a necessary and sufficient condition for robust stability is that none of the q inequalities

$$x \le \|(M)_i\|_{\mathcal{A}} \cdot x \qquad i = 1, \cdots, q$$

has a solution in $(0, \infty)$. Clearly, a necessary and sufficient condition for that to happen is that $\|(M)_i\|_{\mathcal{A}} < 1$ for all i, or equivalently $\|M\|_{\mathcal{A}} < 1$. This is exactly the problem solved by Dahleh and Ohta in [4].

2) Input Sensitivity in the Presence of Multiplicative Input Perturbations: Let P_o be a given nominal linear shift-invariant discrete-time plant with q inputs and p outputs. Consider the following family of plants formed by adding weighted multiplicative perturbations to this nominal plant:

$$\Pi := \{P : P = P_o(I + W_1\Delta), \Delta \in \Delta(q, q)\}$$

where $W_1 \in \mathcal{L}_{TI}^{q \times q}$. Let $S(P_o)$ be defined as follows:

$$S(P_o) := \{C : C \text{ is linear causal shift-invariant controller stabilizing } P_o\}.$$

For a fixed $C \in S(P_o)$ and $\gamma > 0$ we will now obtain necessary and sufficient conditions for C to stabilize every $P \in \Pi$, and at the same time satisfy $\|(I + CP)^{-1}W_2\| < \gamma$ for all P in Π. Hence, the performance objective in this case is keeping small the norm of the weighted input sensitivity function $(I + CP)^{-1}W_2$ despite the presence of the multiplicative perturbations.

This problem can be set up in the framework discussed in the previous sections where a fictitious perturbation replaces the performance objective, thus transforming this stability and performance robustness problem into a stability robustness problem alone. This alternate problem has $\mathcal{D}[(q, q), (q, q)]$ as the class of perturbations, and an M matrix of the following form:

$$M = \begin{pmatrix} \frac{1}{\gamma}(I + CP_o)^{-1}W_2 & CP_o(I + CP_o)^{-1}W_1 \\ \frac{1}{\gamma}(I + CP_o)^{-1}W_2 & CP_o(I + CP_o)^{-1}W_1 \end{pmatrix}.$$

From Table I and Theorem 4, necessary and sufficient conditions for robust stability for this problem, and hence, for robust performance for the original one are as follows:

- $\|(T_o)_i\|_{\mathcal{A}} < 1 \qquad i = 1, \cdots, q,$
- $\left\|\frac{1}{\gamma}(S_o)_i\right\|_{\mathcal{A}} + \dfrac{\|(T_o)_i\|_{\mathcal{A}}\left\|\frac{1}{\gamma}(S_o)_j\right\|_{\mathcal{A}}}{1 - \|(T_o)_j\|_{\mathcal{A}}} < 1 \qquad i, j = 1, \cdots, q.$

where $S_o := (I + CP_o)^{-1}W_2$, and $T_o := CP_o(I + CP)^{-1}{}_oW_1$. Equivalently, these conditions can be written as the following:

- $\|(T_o)_i\|_{\mathscr{A}} < 1 \qquad i = 1, \cdots, q,$
- $\left\|\left(\frac{1}{\gamma}S_o\right)_i\right\|_{\mathscr{A}} < 1 \qquad i = 1, \cdots, q,$
- $\dfrac{\|(T_o)_i\|_{\mathscr{A}}}{\left(1 - \|\frac{1}{\gamma}(S_o)_i\|_{\mathscr{A}}\right)} \cdot \dfrac{\|\frac{1}{\gamma}(S_o)_j\|_{\mathscr{A}}}{(1 - \|(T_o)_j\|_{\mathscr{A}})} < 1 \qquad i, j = 1, \cdots, q$

which, in turn, are equivalent to the following conditions:

- $\|T_o\|_{\mathscr{A}} < 1,$
- $\max_{1 \le i \le q} \dfrac{\|(S_o)_i\|_{\mathscr{A}}}{1 - \|(T_o)_i\|_{\mathscr{A}}} < \gamma.$

If we define $\Psi := \{C \in S(P_o): C \text{ stabilizes all } P \in \Pi\}$, then it follows from our stability robustness conditions for one Δ that $C \in \Psi$ if and only if $C \in S(P_o)$ and $\|T_o\|_{\mathscr{A}} < 1$. Hence, we have shown through the two conditions obtained above that for any $C \in \Psi$

$$\sup_{P \in \Pi} \|(I + CP)^{-1}W_2\| = \max_i \frac{\|(S_o)_i\|_{\mathscr{A}}}{1 - \|(T_o)_i\|_{\mathscr{A}}}.$$

This is exactly the result obtained by the authors in [5] using a different approach. In fact, it is not difficult to show [5] that for any $\gamma > 0$

$$C \in \Psi \text{ and } \sup_{P \in \Pi} \|(I + CP)^{-1}W_2\| < \gamma \Leftrightarrow C \in S(P_o) \text{ and } \|(S_o \quad \gamma T_o)\|_{\mathscr{A}} < \gamma.$$

Since it is known [2], [8], [9] how to solve problems like

$$\min_{C \in S(P_o)} \|(S_o \quad \gamma T_o)\|_{\mathscr{A}}$$

it is clear how an iterative scheme can be devised whereby the value of γ can be increased or decreased according to the outcome of the optimization problem stated previously, until γ is as close as desired to γ_{opt}, where

$$\gamma_{opt} := \inf_{C \in \Psi} \sup_{P \in \Pi} \|(I + CP)^{-1}W_2\|.$$

Since at each iteration step a controller that achieves the minimum can be computed, we can find a controller that achieves arbitrarily closely γ_{opt}.

3) Output Sensitivity in the Presence of Output Multiplicative Perturbations: For this case let

$$\Pi := \{P : P = (I + \Delta W_1)P_o, \Delta \in \Delta(q, q)\}$$

where P_o and W_1 are as before. Suppose we are now interested in the norm of the output sensitivity function as a performance measure. For $C \in S(P_o)$, the M matrix now has the form

$$M = \begin{pmatrix} \frac{1}{\gamma}W_2(I + P_oC)^{-1} & \frac{1}{\gamma}W_2(I + P_oC)^{-1} \\ W_1P_oC(I + P_oC)^{-1} & W_1P_oC(I + P_oC)^{-1} \end{pmatrix}.$$

Hence, from Table I necessary and sufficient conditions for robust stability and performance are now:

- $\|(T_o)_i\|_{\mathscr{A}} < 1 \qquad i = 1, \cdots, q$
- $\|\frac{1}{\gamma}(S_o)_i\|_{\mathscr{A}} + \dfrac{\|(T_o)_j\|_{\mathscr{A}}\|\frac{1}{\gamma}(S_o)_i\|_{\mathscr{A}}}{1 - \|(T_o)_j\|_{\mathscr{A}}} < 1 \qquad i, j = 1, \cdots, q$

where $T_o := W_1P_oC(I + P_oC)^{-1}$ and $S_o := W_2(I + P_oC)^{-1}$. Equivalently, these conditions can be written as follows:

- $\|T_o\|_{\mathscr{A}} < 1$
- $\dfrac{\|S_o\|_{\mathscr{A}}}{1 - \|T_o\|_{\mathscr{A}}} < \gamma.$

With Ψ defined as before, it follows that

$$\text{for any } C \in \Psi, \ \sup_{P \in \Pi} \|W_2(I + PC)^{-1}\| = \frac{\|S_o\|_{\mathscr{A}}}{1 - \|T_o\|_{\mathscr{A}}}.$$

Even though these conditions are different from those obtained in the input sensitivity case, for a scalar plant they are actually the same.

4) Sensitivity Minimization in the Presence of Additive and Multiplicative Perturbations: So far all the examples considered involved at most two Δ's. We now look at an example where three Δ's enter the analysis ($n = 3$). Despite the fact that given a specific $C \in S(P_o)$ the conditions obtained for robust stability/performance are relatively simple to test and hence are ideal for analysis purposes for large values of n, in general, designing a controller that achieves robust stability and performance is not as simple a problem especially when $n > 2$. This becomes apparent when looking at the conditions for $n = 3$ in Table I. However, as this example demonstrates, in some important applications we can exploit the structure of the specific problem at hand to reduce these apparently complex conditions into simple ones which lend themselves easily to optimization procedures, thus facilitating synthesis. To see this, consider the class of plants formed by adding weighted additive as well as multiplicative perturbations to a nominal plant P_o. Multiplicative perturbations represent, for example, unmodeled high-frequency dynamics, sensor noise, etc., whereas the additive perturbations represent the unmodeled time variations in the plant and nonlinear part remaining after linearizing a nonlinear plant about an operating point. For simplicity, we shall look at the scalar case alone. Hence, let

$$\Pi := \{P : P = P_o + \Delta_2W_2P_o + \Delta_3W_3, \text{ where } \Delta_2, \Delta_3 \in \Delta(1,1)\}.$$

Here, $W_2, W_3 \in \mathscr{L}_{TI}^{1 \times 1}$ are stable weights. With this class of plants, we now look at the problem of robust output sensitivity minimization subject to robust stability. So it is desired to minimize the worst case value, as P varies over Π, of $\|W_1(I + PC)^{-1}\|$ subject to robust stability. We start by fixing $C \in S(P_o)$ and then finding necessary and sufficient condition for $\|(1/\gamma)W_1(I + PC)^{-1}\| < 1$ for all $P \in \Pi$ subject to robust stability. As before, this problem fits very naturally in our framework and is equivalent to a stability robustness problem with the class of perturbations consisting of

$\mathcal{S}[(1,1);\cdots;(1,1)]$, and an M matrix as follows

$$M = \begin{pmatrix} \frac{1}{\gamma}W_1(I+P_oC)^{-1} & \frac{1}{\gamma}W_1(I+P_oC)^{-1} & \frac{1}{\gamma}W_1(I+P_oC)^{-1} \\ -W_2P_oC(I+P_oC)^{-1} & -W_2P_oC(I+P_oC)^{-1} & -W_2P_oC(I+P_oC)^{-1} \\ -W_3C(I+P_oC)^{-1} & -W_3C(I+P_oC)^{-1} & -W_3C(I+P_oC)^{-1} \end{pmatrix}.$$

Applying Theorem 3, or equivalently looking at Table I for $n = 3$, we find the following necessary and sufficient conditions:

- $$\|R_o\|_{\mathcal{A}} < 1$$
- $$\|T_o\|_{\mathcal{A}} + \frac{\|T_o\|_{\mathcal{A}}\|R_o\|_{\mathcal{A}}}{1-\|R_o\|_{\mathcal{A}}} < 1$$
- $$\left\|\frac{1}{\gamma}S_o\right\|_{\mathcal{A}} + \frac{\left\|\frac{1}{\gamma}S_o\right\|_{\mathcal{A}}\|R_o\|_{\mathcal{A}}}{1-\|R_o\|_{\mathcal{A}}} + \frac{\left(\left\|\frac{1}{\gamma}S_o\right\|_{\mathcal{A}} + \frac{\left\|\frac{1}{\gamma}S_o\right\|_{\mathcal{A}}\|R_o\|_{\mathcal{A}}}{1-\|R_o\|_{\mathcal{A}}}\right)\left(\|T_o\|_{\mathcal{A}} + \frac{\|T_o\|_{\mathcal{A}}\|R_o\|_{\mathcal{A}}}{1-\|R_o\|_{\mathcal{A}}}\right)}{1-\left(\|T_o\|_{\mathcal{A}} + \frac{\|T_o\|_{\mathcal{A}}\|R_o\|_{\mathcal{A}}}{1-\|R_o\|_{\mathcal{A}}}\right)} < 1$$

where $S_o := W_1(I+P_oC)^{-1}$, $T_o := W_2P_oC(I+P_oC)^{-1}$, and $R_o := W_3C(I+P_oC)^{-1}$. These three conditions are equivalent to the following condition:

- $$\left\|\frac{1}{\gamma}S_o\right\|_{\mathcal{A}} + \|T_o\|_{\mathcal{A}} + \|R_o\|_{\mathcal{A}} < 1.$$

This condition is the key to controller synthesis. As before, this can be done by iteration on γ, and we can get as close as desired to the optimal γ.

Finally, we use the aforementioned three conditions to derive an explicit expression for the quantity $\sup_{P\in\Pi}\|W_1(I+PC)^{-1}\|$ for any $C \in \Psi$. This is done by observing that the first two conditions are exactly those needed for robust stability alone and can be rewritten as the following:

- $$\|T_o\|_{\mathcal{A}} + \|R_o\|_{\mathcal{A}} < 1.$$

Simplifying the third condition above and combining it with this one, we can write

$$C \in \Psi \text{ and } \|W_1(I+PC)^{-1}\| < \gamma \qquad \text{for all } P \in \Pi$$

$$C \in S(P_o) \text{ and } \|T_o\|_{\mathcal{A}} + \|R_o\|_{\mathcal{A}} < 1$$

$$\text{and } \frac{\|S_o\|_{\mathcal{A}}}{1-(\|T_o\|_{\mathcal{A}} + \|R_o\|_{\mathcal{A}})} < \gamma.$$

It follows that for any $C \in \Psi$

$$\sup_{P\in\Pi}\|W_1(I+PC)^{-1}\| = \frac{\|S_o\|_{\mathcal{A}}}{1-(\|T_o\|_{\mathcal{A}} + \|R_o\|_{\mathcal{A}})}.$$

This expression can be used for analysis purposes, after the condition for robust stability has been checked. It is interesting to see how robust stability cannot be separated from robust performance since without robust stability the expression for the worst case performance makes no sense.

We next provide a numerical example demonstrating the advantages of incorporating performance robustness considerations in the design procedure.

Example 1: Assume a physical plant is modeled by the following plant class:

$$\Pi_r := \{P = (I + \Delta r W_1)P_o : \Delta \in \Delta(1,1)\}$$

where $P_o = z(z-0.1)/(z-0.5)(z-2)$,[2] $W_1 = 0.1/z + 1.1$, and r is a positive real number representing the assumed radius of perturbation ball. This class is the same as that considered in the third application example in this section, with the only difference being that here we show the dependence on r explicitly instead of absorbing it in the weight W_1.

We have shown in this section that $C \in S(P_o)$ achieves robust stability if and only if $\|rT_o\|_{\mathcal{A}} < 1$, and that for a robustly stabilizing controller, the worst case norm of weighted sensitivity function is given by

$$\sup_{P\in\Pi_r}\|(I+PC)^{-1}W_2\| = \frac{\|S_o\|_{\mathcal{A}}}{1-\|rT_o\|_{\mathcal{A}}}$$

where $S_o := (I+P_oC)^{-1}W_2$ and $T_o := P_oC(I+P_oC)^{-1}W_1$. For this example, we choose $W_2 = 0.5/z - 5.0$. Before attempting any design procedure, we can compute the maximum perturbation ball radius that can be tolerated without violating robust stability, regardless of the choice of controller in $S(P_o)$. This number is equal to $1/\min_{C\in S(P_o)}\|T_o\|_{\mathcal{A}} := r_{\max}$. For our specific problem data, $r_{\max} = 2.90909$. If $r \geq r_{\max}$, robust stability is lost and no controller in $S(P_o)$ can restore it. In fact, the results in [6] show that even allowing the controller to be time varying does not help. We therefore restrict ourselves to $r < r_{\max}$. Next, we compare the robustness properties of three design procedures. The first of these ignores the perturba-

[2] We adopt the convention that the $\mathcal{Z}$ transform of a signal u is $\sum_{k=0}^{\infty}u(k)z^k$.

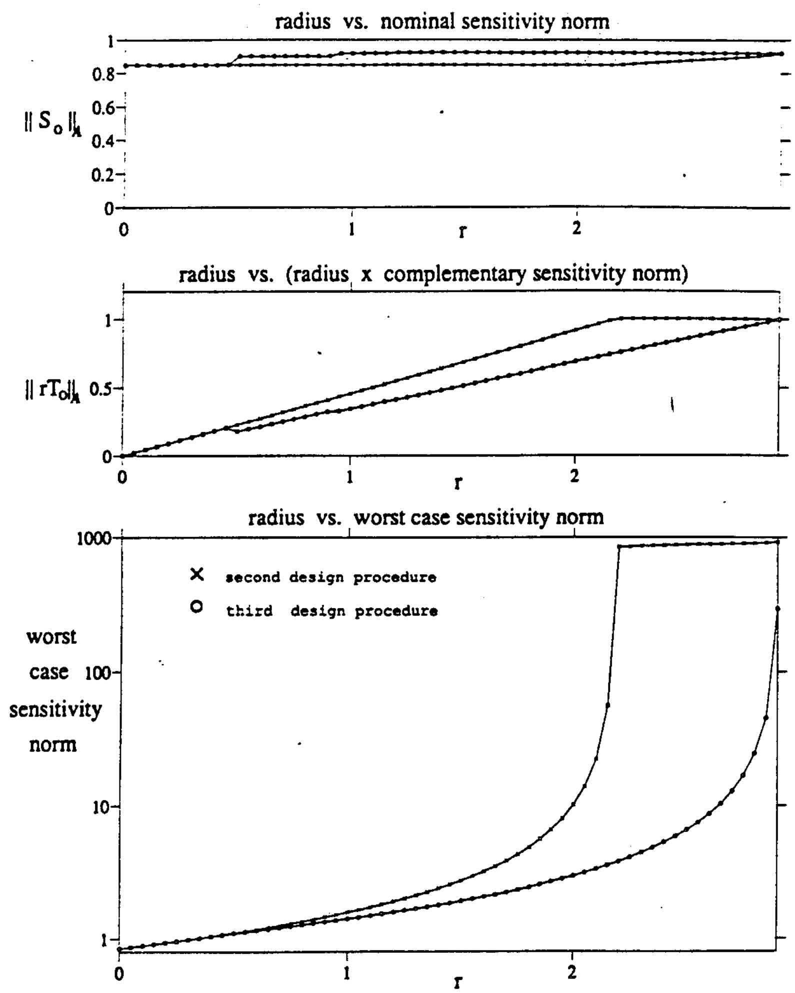


Fig. 4. Second and third design procedures in Example 1.

tions and minimizes the norm of the weighted nominal sensitivity function. The second design procedure does the same but with the added constraint that robust stability is to be maintained, while the third procedure minimizes the *worst case* norm of the weighted sensitivity function subject to robust stability. We consider each of these separately.

In the first procedure, we solve using the techniques in [2], [8] the problem

$$\min_{C\in S(P_O)} \|(I + P_oC)^{-1}W_2\| := \gamma_{\text{nom}}.$$

For our example, $\gamma_{\text{nom}} = 0.7265306$. Computing the Q parameter associated with γ_{nom}, we can find the corresponding T_o whose $\mathscr{A}$ norm turns out to be 7.0979592. Therefore, robust stability is achieved only for $r < 1/\|T_o\|_{\mathscr{A}} = 0.14088557$, which is much less than the maximum achievable value of $r_{\max}$. Furthermore, for this range of r

$$\sup_{P\in\Pi_r} \|(I + PC)^{-1}W_2\| = \frac{\|S_o\|_{\mathscr{A}}}{1 - \|rT_o\|_{\mathscr{A}}} = \frac{0.7265306}{1 - 7.0979592\, r}$$

which approaches ∞ as r approaches 0.14088557. Notice that the design scheme does not depend on r since it completely ignores the perturbations.

The second design procedure attempts to achieve robust stability for larger values of r by solving the problem

$$\min_{C\in S(P_O)} \|S_o\|_{\mathscr{A}}$$
$$\text{subject to } \|rT_o\|_{\mathscr{A}} < 1.$$

In its present form, this problem has no solution. We shall solve the following slightly modified form of it which does have a solution:

$$\min_{C\in S(P_O)} \|S_o\|_{\mathscr{A}}$$
$$\text{subject to } \|rT_o\|_{\mathscr{A}} \le 1 - \epsilon$$

where $\epsilon > 0$. For our example, we shall pick $\epsilon = 0.001$ and solve this problem for various possible values of r. Fig. 4 shows the resulting values of $\|S_o\|_{\mathscr{A}}$, $\|rT_o\|_{\mathscr{A}}$ and $\sup_{P\in\Pi_r}\|(I + PC)^{-1}W_2\|$ as functions of r. Of course for an actual design, the value of r is chosen *a priori* and the optimization problem is solved for that particular r. The numbers appearing in the figure were obtained by solving the previous optimization problem over all polynomial closed-loop objective functions with degree less than or equal to 11 (see [2]; [8] for more details on solving truncated problems). As may be seen in the figure, even though this design method acknowledges the existence of the perturbations and as a result yields systems which are robustly stable for

values of r as large as $r_{\max}$, these designs suffer from extremely bad performance robustness properties, especially for $r > 2$. In fact, for $r > 2.18$, the worst-case norm of the nominal sensitivity approaches 1000! Worse still, if one attempts to further decrease the norm of the nominal sensitivity by making ϵ smaller but still keeping it larger than zero to guarantee robust stability, the worst case norm of the sensitivity function gets much larger despite the smaller value for the nominal sensitivity norm. It can be made arbitrarily large by making ϵ sufficiently small.

Fortunately, the third design scheme does not suffer from any of the problems associated with the first two design schemes. It is based on solving the following problem:

$$\inf_{C \in \Psi} \sup_{P \in \Pi_r} \|(I + PC)^{-1} W_2\| = \inf_{\substack{C \in S(P_o) \\ \|rT_o\|_{\mathscr{A}} < 1}} \frac{\|S_o\|_{\mathscr{A}}}{1 - \|rT_o\|_{\mathscr{A}}}.$$

Fig. 4 shows the resulting values of $\|S_o\|_{\mathscr{A}}$, $\|rT_o\|_{\mathscr{A}}$ and $\sup_{P \in \Pi_r} \|(I + PC)^{-1} W_2\|$ for various assumed values of r. To allow comparison with the second scheme, a maximum closed-loop polynomial degree of 11 was used here as well. The figure indicates that this design scheme not only has much better performance robustness properties than the first two schemes, but that it also has superior stability robustness properties as shown by the values of $\|rT_o\|_{\mathscr{A}}$. This means that it can tolerate, without losing stablity, perturbations with radius even larger than r, the perturbation radius which was used in the design. These large improvements in both stability and performance robustness properties are gained at the very small cost of a slight increase in the norm of the nominal sensitivity function.

IX. Conclusion

We have provided in the previous sections necessary and sufficient conditions for achieving stability and performance robustness. These conditions can be applied to a large class of problems in which multiple perturbations can enter in various configurations. The conditions involve no more than computing the $\mathscr{A}$ norm of certain transfer functions, a task which can be done to any degree of accuracy with relative ease. Consequently, these conditions provide a particularly attractive method for the analysis of stability and performance robustness. We have also shown that in some important cases obtaining a controller with optimal robustness properties can be done through a simple iterative scheme. Synthesis of controllers in the more general case, is an interesting problem which is currently under research.

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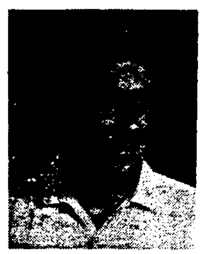

Mustafa H. Khammash was born in 1964. He received the B.S. degree from Texas A&M University, College Station, in 1986 and the Ph. D. degree from Rice University, Houston, TX, in 1990, both in electrical engineering.

He is currently an Assistant Professor of Electrical Engineering at Iowa State University, Ames.

Dr. Khammash is the recipient of the 1990 Ralph Budd Award for outstanding engineering Ph. D. thesis. He is a member of Eta Kappa Nu and Tau Beta Pi.

J. Boyd Pearson, Jr., (S'57–M'58–M'65–SM'77–F'79) received the Ph.D. degree in 1962 from Purdue University, Lafayette, IN.

He served as Assistant Professor of Electrical Engineering at Purdue from 1962 to 1965, when he joined Rice University, Houston, TX, where he is now a J.S. Abercrombie Professor.

Dr. Pearson served as President of the IEEE Control Systems Society in 1984 and received outstanding paper awards in 1985 with Bor-Chin Chang and in 1989 with Munther Dahleh.

A General Framework for Linear Periodic Systems with Applications to $\mathcal{H}^\infty$ Sampled-Data Control

Bassam A. Bamieh, *Member, IEEE*, and J. Boyd Pearson, Jr., *Fellow, IEEE*

Abstract—**We present a framework for dealing with continuous-time periodic systems. The main tool is a lifting technique which provides a strong correspondence between continuous-time periodic systems and certain types of discrete-time time-invariant systems with infinite dimensional input and output spaces. Despite the infinite dimensionality of the input and output spaces, a lifted system has a finite-dimensional state space if the original system does. This fact permits rather constructive methods for analyzing these systems. As a demonstration of the utility of this framework, we use it to describe the continuous time (i.e., intersample) behavior of sampled-data systems, and to obtain a complete solution to the problem of parametrizing all controllers that constrain the L^2-induced norm of a sampled-data system to within a certain bound.**

Introduction

Our motivation for studying continuous-time periodic systems comes from considering sampled-data control systems, in which a discrete-time controller is used in feedback with a continuous-time plant. The interconnection between the two parts of the system is typically through sample and hold devices. In most treatments of sampled-data systems, the continuous-time plant is in some way discretized, and one designs a controller for the discretized plant. Generally, this treatment describes the behavior of the overall system only at the sampling instants, and the intersample behavior is lost in the process of discretization.

Recently, there has been an increased interest in problems involving the intersample behavior of sampled-data systems. The impetus for this comes from robust control problems for which it is more natural to consider the sampled-data system in continuous time. For example, in the disturbance rejection problem, since the physical system being controlled (the plant) evolves in continuous time, it is reasonable to consider the disturbances as continuous-time signals. When measuring the effect of disturbances on other signals in the system, this has to be done at all times (i.e., in between samples), and not only at the sampling instants. Another example is given by the robust stability problem (in $\mathcal{H}^\infty$), in which the uncertainty in the plant is described as a weighted error bound on the frequency response of a nominal plant. The resulting perturbation (on the nominal plant), is a continuous-time system, therefore, if one is to use, for example, the small gain theorem to stabilize the whole family of plants, one must consider norms in the system over continuous time.

In this paper, we will be concerned with the problem of bounding the L^2-induced norm (in continuous time) of sampled-data systems. The setup is shown in Fig. 1, where G is a continuous-time time-invariant generalized plant, C is discrete-time time-invariant, $\mathcal{H}_\tau$ is a zero-order hold (with period τ), and $\mathcal{S}_\tau$ is an ideal sampler (with period τ). $\mathcal{H}_\tau$ and $\mathcal{S}_\tau$ are assumed to be synchronized, they provide the interface between the digital and the analog parts of the system. We call $\mathcal{H}_\tau C \mathcal{S}_\tau$ the sampled-data controller. The exogenous input w contains disturbances and command signals, the regulated output z are the variables which should be made "small," note that they are both continuous-time signals, since we want to describe the input-output behavior of the sampled-data system in continuous time. We also call the arrangement in Fig. 1 the hybrid system, to emphasize that we are considering the overall behavior of the system.

The problem we consider is given $\gamma > 0$, to find C such that the L^2 induced norm of the mapping from w to z is less than γ. We call this, the *standard problem with sampled-data controllers* (or the sampled-data problem for short). The difference between this problem and the so-called "standard problem" is that in the latter, if G is a continuous-time time-invariant system, then only continuous-time time-invariant controllers are considered. In our problem, the *continuous-time* controller is constrained to be a sampled-data controller, i.e., it is of the form $\mathcal{H}_\tau C \mathcal{S}_\tau$, where C is a discrete-time system.

The standard problem with sampled-data controllers is significantly different from the usual standard problem, three major differences are as follows.

i) There is a "structural constraint" on the controller, that is, it is constrained to be of the form $\mathcal{H}_\tau C \mathcal{S}_\tau$.

ii) The controller $\mathcal{H}_\tau C \mathcal{S}_\tau$ is not time invariant even if C is time invariant (in discrete time). Therefore, even if G is also time invariant, the overall system in Fig. 1 is time varying, in fact, it is periodic, with period τ (τ-periodic).

iii) The hybrid nature of the system is problematic, since not all parts of the system are defined over the same time set.

In this paper, we present a framework for periodic systems

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B. A. Bamieh is with the Department of Electrical and Computer Engineering and Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801.

J. B. Pearson, Jr. is with the Department of Electrical and Computer Engineering, Rice University, Houston, TX 77251.

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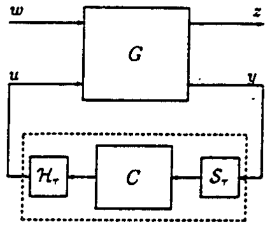

Fig. 1. A continuous-time G with a sampled-data controller.

in continuous time. We then apply this framework to the sampled-data problem and show that it provides a satisfactory answer to the three difficulties mentioned above, and we obtain a complete solution to the problem. To briefly describe the solution, let us use the notation $\mathscr{F}(P, K)$ to denote the closed-loop mapping (from exogeneous input to regulated output) of a generalized plant P in feedback with K. In this notation the closed-loop mapping in Fig. 1 is given by $\mathscr{F}(G, \mathscr{H}_\tau C \mathscr{S}_\tau)$. The solution of the sampled-data problem is given in terms of an "equivalent" discrete-time time-invariant generalized plant $\acute{G}$ such that

$$\|\mathscr{F}(G, \mathscr{H}_\tau C \mathscr{S}_\tau)\| < \gamma \Leftrightarrow \|\mathscr{F}(\acute{G}, C)\| < \gamma \tag{1}$$

where $\|\mathscr{F}(G, \mathscr{H}_\tau C \mathscr{S}_\tau)\|$ is the L^2-induced norm, and $\|\mathscr{F}(\acute{G}, C)\|$ is the $\mathscr{H}^\infty$ norm. Therefore, once $\acute{G}$ is found, the sampled-data problem is equivalent to a discrete-time $\mathscr{H}^\infty$ problem, the solution of the latter is well known [16]. In a sense, $\acute{G}$ is a "discretization" of G, but it is an equivalent discretization for norm problems, in that the induced norms satisfy (1).

As already mentioned, the solution just described is obtained using a framework that we develop for periodic systems in continuous time. The main tool in this framework is a *lifting* technique similar to that used for discrete-time periodic systems in [19]. There are however, considerable differences between the lifting techniques in discrete time and continuous time, respectively. In discrete time, an N-periodic system is lifted to a time-invariant multivariable system (of larger input–output dimensions). In continuous time, the appropriate lifting takes vector-valued signals to signals which take values in a general Banach space as opposed to a finite-dimensional space, as a result, the time-invariant lifted systems have infinite-dimensional input and output spaces, and the theory is more technical, but many of the desirable features of the lifting remain true. For example, the lifting preserves algebraic operations on systems and the norms of signals and systems. Another crucial point is that our lifted systems will have *finite*-dimensional state spaces if the original systems are finite-dimensional. Since the lifted systems have infinite-dimensional input and output spaces, we will abuse terminology by calling them (for lack of a better term) infinite-dimensional, even though this term is better reserved for systems which have an infinite-dimensional state space.

This paper is organized as follows: in Section I, we introduce the lifting technique and show that it provides a strong correspondence between continuous-time periodic systems and certain types of discrete-time time-invariant infinite-dimensional systems. In Section II, we study further the time-invariant infinite-dimensional systems, in particular, their z-transforms and state-space realizations. Not all the material covered in these two sections is essential for later developments, but it is included to provide a more complete discussion. In Section III, the lifting technique is applied to the hybrid system of Fig. 1 to convert the sampled-data problem into an equivalent infinite-dimensional standard problem where the generalized plant has a finite-dimensional state space. Up to this point the discussion covers all L^p-induced norm problems, we then specialize to the L^2-induced norm case, and in Section IV, the finite-dimensionality of the state-space models is exploited to reduce the infinite-dimensional standard problem to a finite-dimensional standard $\mathscr{H}^\infty$ problem. The main theorem (6) provides the equivalent generalized plant $\acute{G}$, explicit formulas for the state-space description are derived in Section V directly in terms of the state-space description of the original plant G.

We now comment briefly on some of the related recent work on sampled-data systems. In [3] solutions were obtained to problems where the induced norm is from a discrete-time input to a continuous-time output and vice versa. In [20] a characterization was given for the L^2-induced norm of a sampled-data system assuming ideally band limited input signals.

The works which are most related to ours are [14], [17], [18], [25], [26]. In [14], [17], a solution was announced (though derivations were not given) to the $\mathscr{H}^\infty$ sampled-data problem that is similar to our solution, that is, the norm of the sampled-data system is equivalent to the norm of a discrete-time system. Reference [26] is related to our work in terms of the technique used, the paper does not address norm problems, but a tracking problem. In [26] a lifting technique is developed which is equivalent to the one developed in this paper but with an important difference, in [26] the lifted systems are realized with an infinite-dimensional state-space, while as will be seen in this paper, the finite-dimensionality of the state space (of lifted systems) is the crucial fact that solves the $\mathscr{H}^\infty$ problem.

While this paper was being reviewed, we received [18], [25]. Reference [18] contains the derivation of the results announced in [14], [17], the technique used there consists of forming a Hamiltonian which characterizes the norm of the sampled-data system, this is different from our lifting technique, although it is interesting that similar final results are obtained. The work in [25] is remarkably similar to ours, the author uses an equivalent of the lifting technique (although in [25] it is not called as such) and obtains a problem with infinite-dimensional input and output spaces and finite-dimensional state space, also the reduction to a standard $\mathscr{H}^\infty$ problem is done in a similar way to ours. The one exception to this similarity is that the reduction is not done completely (the missing step in [25] is Lemma 5 in this paper), thus the equivalence in [25] is approximate (to any degree of accuracy). In contrast, the equivalence in (1) is exact.

In [1], [2] ([24] announces similar results), a somewhat different approach to the sampled-data problem is taken. There, the problem is posed where sampled measurements

are available and the optimum control is to be found for all time (including in between the samples), and one obtains time-varying controllers. The interesting contrast is that one obtains the optimum waveforms of the control signals in between the samples, while in our setup, the control signal is constrained to be constant on sampling intervals. Thus the performance of the controllers in [1], [2], [24], is in general better than the ones given here, since a wider class of controllers is allowed. But it is to be noted that the two problems are distinct, in that the use of a sampled-data controller obviously puts more constraints on the problem. It would be interesting to compare the two problems and quantify the loss of performance that results from using a zero-order hold.

Finally, we note that the framework for continuous-time periodic systems presented in this paper was independently developed by B. A. Francis and A. Tannenbaum.

I. The Lifting Technique for Continuous-Time Periodic Systems

The lifting is first defined for signals, this definition, in turn, induces a definition of the lifting for systems. It turns out that to convert the periodicity of a system to a time-invariance of its lifting, the lifting must be such that continuous-time signals are lifted to discrete-time signals that take their values in a function space (see Fig. 2). In order to do this systematically we need to define the appropriate signal spaces.

Let us begin with the usual signal spaces in continuous-time $L_N^p[0,\infty)$, $1 \le p \le \infty$, and the extended signal spaces $L_{N,e}^p[0,\infty)$, $1 \le p \le \infty$. The signals can be N-vectors of scalar signals. To avoid cumbersome notation, the dimensions of signals and systems will be omitted from now on, and we write L^p instead of L_N^p. We adopt the notation that a statement involving L^p or L_e^p without assigning a value for p, is referring to any p. Also L^p (L_e^p) will denote $L^p[0,\infty)(L_e^p[0,\infty))$ when no confusion can occur.

To introduce the lifting, we first need to define spaces of vector valued signals. From now on, by vector valued we mean Banach space valued. For any Banach space X, let l_X be the space of sequences which take values in X, that is $\{f_i\}: \mathbb{N} \to X$. We use the notation that $\{f_i\}$ is a sequence, each element of which is f_i, so

$$l_X = \{\{f_i\}, f_i \in X \ \forall i\}.$$

Note that this is consistent with the notation l_N^p for signals that are sequences of N-vectors, that is, the signals that take values in a finite-dimensional Banach space $\mathbb{R}^N$.

Norms can be added to these spaces by considering the l_X^p spaces:

$$l_X^p = \left\{\{f_i\} \in l_X;\ \left(\sum_{i=0}^{\infty} \|f_i\|_X^p\right)^{1/p} < \infty\right\}, \qquad 1 \le p < \infty$$

$$l_X^\infty = \left\{\{f_i\} \in l_X;\ \sup_i \|f_i\| X < \infty\right\}.$$

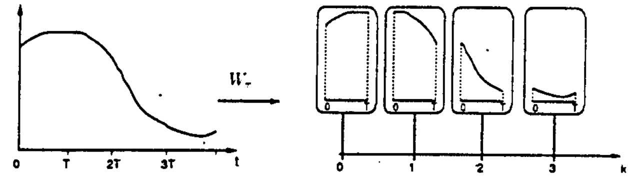

Fig. 2. $W_\tau: L_e^p[0,\infty) \to l_{L^p[0,\tau]}$.

The norms are given by

$$\|\{f_i\}\|_{l_X^p} = \left(\sum_{i=0}^{\infty} \|f_i\|_X^p\right)^{1/p},$$

$$1 \le p < \infty, \ \|\{f_i\}\|_{l_X^\infty} = \sup_i \|f_i\|_X.$$

It can be shown that with these norms, l_X^p are indeed Banach spaces [4, III.4.4]. We shall be particularly interested in signals in $l_{L^p[0,\tau]}$ and $l_{L^p[0,\tau]}^p$, these can be visualized as sequences $\{f_i\}$, or discrete-time signals which, for each time i take values which are functions in $L^p[0,\tau]$. We are now ready to define the lifting for each τ, let $W_\tau: L_e^p[0,\infty) \to l_{L^p[0,\tau]}$ be defined by

$$\hat{f} = W_\tau f, \quad \hat{f}_i(t) = f(\tau i + t), \qquad 0 \le t \le \tau. \quad (2)$$

The definition says that $\hat{f}$ is a sequence, each element of which is a function of t, $0 \le t \le \tau$ given by (2). Since $f \in L_e^p[0,\infty)$, this means that $\hat{f}_i \in L^p[0,\tau]$ for each i, and thus W_τ is well defined. The lifting, W_τ, can be visualized as breaking up the signal f defined on the real line into an infinite number of pieces, each piece is a copy of f restricted to a line segment of length τ, this is illustrated in Fig. 2.

W_τ defined on the linear space $L_e^p[0,\infty)$ is a linear transformation. It also follows that W_τ is one-to-one and onto, thus invertible, this can be seen by explicitly constructing the inverse as follows:

$$f = W_\tau^{-1} g$$

$$f(t) = g_i(t - \tau i), \qquad \text{for } \tau i \le t \le \tau(i+1).$$

This can again be visualized as the reverse of the operation in Fig. 2, W_τ^{-1} takes a sequence of function pieces, each a function in $L^p[0,\tau]$ and "glues" them together in order, thus forming a function $f \in L_e^p[0,\infty)$. W_τ is then linear bijection between $L_e^p[0,\infty)$ and $l_{L^p[0,\tau]}$.

If we restrict the domain of W_τ to the Banach space $L^p[0,\infty) \subset L_e^p[0,\infty)$, we can show that $W_\tau: L^p[0,\infty) \to l_{L^p[0,\tau]}$, and is an isometry between these two Banach spaces. This is a consequence of the following computation:

$$\hat{f} = W_\tau f$$

$$\begin{aligned}
\|\hat{f}\|_{l^p_{L^p[0,\tau]}}^p &= \sum_{i=0}^{\infty} \|\hat{f}_i\|_{L^p[0,\tau]}^p \\
&= \sum_{i=0}^{\infty} \left(\left(\int_0^\tau |\hat{f}_i(t)|^p \, dt\right)^{1/p}\right)^p \\
&= \sum_{i=0}^{\infty} \int_0^\tau |f(\tau i + t)|^p \, dt \\
&= \int_0^\infty |f(\hat{t})|^p \, d\hat{t} = \|f\|_{L^p[0,\infty)}^p
\end{aligned}$$

for $1 \le p < \infty$. For the case $p = \infty$, we have

$$\begin{aligned}\|f\|_{L^\infty[0,\infty)} &= \operatorname*{ess\,sup}_{0 \le t < \infty} |f(t)| \\ &= \sup_i \left(\operatorname*{ess\,sup}_{0 \le \hat{t} \le \tau} |f(\tau i + \hat{t})| \right) \\ &= \sup_i \|\hat{f}_i\|_{L^\infty[0,\tau]} = \|\hat{f}\|_{l^\infty_{L^\infty[0,\tau]}}.\end{aligned}$$

In summary, W_τ is a bijective linear mapping between $L^p_e[0,\infty)$ and $l_{L^p[0,\tau]}$, and a bijective linear *isometry* between $L^p[0,\infty)$ and $l^p_{L^p[0,\tau]}$.

We now define the lifting for systems. Given any linear operator $G: L^p_e[0,\infty) \to L^p_e[0,\infty)$, let its lifting $\hat{G}: l_{L^p[0,\tau]} \to l_{L^p[0,\tau]}$, be defined by $\hat{G} := W_\tau G W_\tau^{-1}$. By the linearity of each of the defining operators, $\hat{G}$ is linear. Moreover, if G is also bounded $G: L^p[0,\infty) \to L^p[0,\infty)$, then $\hat{G}$ is also bounded and $\hat{G}: l^p_{L^p[0,\tau]} \to l^p_{L^p[0,\tau]}$ since both W_τ and W_τ^{-1} are bounded. The fact that W_τ and W_τ^{-1} are isometries, allows us to make the stronger conclusion that $\|G\| = \|\hat{G}\|$, that is, the system norm is preserved by the lifting.

Algebraic operations are also preserved by the lifting, namely: $(\widehat{G_1 + G_2}) = \hat{G}_1 + \hat{G}_2$ because $W_\tau(G_1 + G_2)W_\tau^{-1} = W_\tau G_1 W_\tau^{-1} + W_\tau G_2 W_\tau^{-1}$, and $(\widehat{G_1 G_2}) = \hat{G}_1 \hat{G}_2$ since $W_\tau G_1 G_2 W_\tau^{-1} = W_\tau G_1 W_\tau^{-1} W_\tau G_2 W_\tau^{-1}$, and if G^{-1} is well defined, $G^{-1}: L^p_e[0,\infty) \to L^p_e[0,\infty)$ then $(\widehat{G^{-1}}) = \hat{G}^{-1}$ because

$$\widehat{G^{-1}}\hat{G} = W_\tau G^{-1} W_\tau^{-1} W_\tau G W_\tau^{-1} = W_\tau I W_\tau^{-1} = I.$$

These properties allow us to conclude that feedback stability is also preserved under lifting. If, by the pair (F, G) being X stable, we mean that the system with F and G in feedback is stable for all exogeneous inputs from the signal space X (i.e., all transfer functions are bounded operators), we can conclude: *the pair (F, G) is $L^p[0,\infty)$ stable if and only if the pair $(\hat{F}, \hat{G})$ is $l^p_{L^p[0,\tau]}$ stable.*

Now if the system to be lifted is τ-periodic, the lifted system should exhibit some sort of time invariance. Let the *delay operator* D_τ be defined as usual by $(D_\tau f)(t) = f(t - \tau)$ for $f \in L^p_e$. Given a system $G: L^p_e \to L^p_e$, we say that G is *τ-periodic* if it commutes with D_τ, that is $D_\tau G = G D_\tau$. G is *time invariant* if $D_\tau G = G D_\tau$ $\forall \tau > 0$. Let S be the right-shift operator defined on any space of sequences, that is, $S(\{x_0, x_1, \cdots\}) = \{0, x_0, x_1, \cdots,\}$ for any sequence $\{x_i\}$, in particular S is defined on any l_X. Another important property of the lifting (of signals) is that it "intertwines" the D_τ and the S operators, that is

$$W_\tau D_\tau = S W_\tau.$$

This intertwining property will convert the commutation with D_τ property of periodic systems into a time-invariance property. To see this, let G be a τ-periodic system, then

$$\begin{aligned}\hat{G}S &= W_\tau G W_\tau^{-1} S = W_\tau G D_\tau W_\tau^{-1} = W_\tau D_\tau G W_\tau^{-1} \\ &= S W_\tau G W_\tau^{-1} = S\hat{G}\end{aligned}$$

that is, $\hat{G}$ commutes with the shift, which can be taken as a definition of time invariance, or rather shift invariance in this general setting.

The above $\hat{G}$ also has a certain type of convolution representation. It can be shown in general that given any Banach space X, and a linear operator F on the sequence space l_X which commutes with the shift, i.e., $FS = SF$, the operator F has the following representation:

$$y = Fu, \quad y_n = (Fu)_n = \sum_{m=0}^{n} F_{n-m}(u_m), \quad F_i \in \mathcal{L}(X, X).$$

That is, F is represented by a sequence $\{F_i\}$ (its "impulse response") where each $F_i \in \mathcal{L}(X, X)$, the space of linear operators on X. This is consistent with the convolution representation of multivariable systems where y_n and u_n are vectors in $\mathbb{R}^N$ and $F_i \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$, i.e., an $N \times N$ matrix. We shall not prove this general characterization, instead, for the particular case where $F = \hat{G}$, a lifting of a periodic system, we will explicitly construct the sequence $\{\hat{G}_i\}$.

To perform the explicit construction, let us assume that the time varying systems involved have a kernel representation, that is, if $G: L^p_e \to L^p_e$ is time varying, it is associated with a kernel $g(t, s)$ such that

$$y = Gu, \quad y(t) = \int_0^t g(t, s) u(s)\, ds \tag{3}$$

where the kernel function is a distribution (in the variable s) [5] of the form

$$g(t, s) = \sum_{i=0}^{\infty} \gamma_i \delta(t - s - h_i) + \tilde{g}(t, s).$$

An assumption that guarantees the existence of the integral for any $u \in L^p_e$ and each t, is that the function $\tilde{g}(t, s)$ be bounded on bounded subsets of $\mathbb{R}^2$, and that the sequence of nonnegative real numbers $\{h_i\}$ be discrete (i.e., have no cluster points).

The scaled identity operator γI is given by the kernel $\gamma\delta(t - s)$, and for the delay operator D_τ, $D_\tau(t, s) = \delta(t - s - \tau)$. Given G, a τ-periodic system, we have by definition $D_\tau G = G D_\tau$, it is easy to show that this is true if and only if the kernel of G has the "block Toeplitz" structure

$$G(t, s) = G(t + \tau, s + \tau). \tag{4}$$

By repeated applications of (4), we derive the following identities to be used later:

$$\begin{aligned}G(t, s) &= G(t + n\tau, s + n\tau) \qquad \text{for } n \ge 0, \\ G(t + i\tau, s + j\tau) &= G(t + (i - j)\tau, s) \qquad \text{for } i \ge j.\end{aligned}$$

Since $G(t, s)$ is block Toeplitz, it is completely determined by a sequence of "blocks," define

$$\hat{G}_i(\hat{t}, \hat{s}) := G(\hat{t} + \tau i, \hat{s}) \qquad \text{for } 0 \le \hat{t} < \tau,\ 0 \le \hat{s} < \tau,\ i > 0.$$

This decomposition is illustrated in Fig. 3. Each $\hat{G}_i(\hat{t}, \hat{s})$ can be regarded as a function on the square $[0, \tau] \times [0, \tau]$ and as such, representing an operator on $L^p[0, \tau]$ as follows, for $\hat{u}, \hat{y} \in L^p[0, \tau]$

$$\hat{y} = \hat{G}_i \hat{u}, \quad \hat{y}(\hat{t}) = \int_0^\tau \hat{G}_i(\hat{t}, \hat{s}) \hat{u}(\hat{s})\, d\hat{s}, \qquad 0 \le \hat{t} < \tau.$$

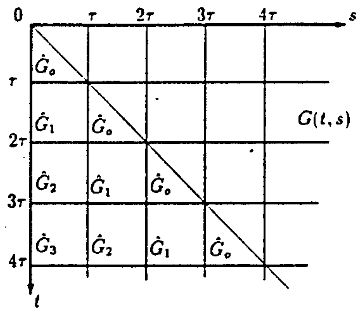

Fig. 3. Block Toeplitz structure of a periodic kernel.

This suggests that $\{\hat{G}_i\}$ can be regarded as the impulse response of the lifting of G, the following computation confirms; for $y, u \in L_e^p[0, \infty)$ let $y = Gu$, then lift the signals

$$\{\hat{y}_i\} = W_\tau y, \quad \{\hat{u}_i\} = W_\tau u$$

then the lifted signals are related by

$$\begin{aligned}
\hat{y}_i(\hat{t}) = y(\tau i + \hat{t}) &= \int_0^\infty G(\tau i + \hat{t}, s)u(s)\,ds \\
&= \sum_{j=0}^{i} \int_{\tau j}^{\tau(j+1)} G(\tau i + \hat{t}, s)u(s)\,ds \\
&= \sum_{j=0}^{i} \int_0^\tau G(\tau i + \hat{t}, \tau j + \hat{s})u(\tau j + \hat{s})\,d\hat{s} \\
&= \sum_{j=0}^{i} \int_0^\tau G(\tau(i-j) + \hat{t}, \hat{s})\hat{u}_j(\hat{s})\,d\hat{s} \\
&= \sum_{j=0}^{i} \int_0^\tau \hat{G}_{i-j}(\hat{t}, \hat{s})\hat{u}_j(\hat{s})\,d\hat{s}
\end{aligned}$$

which can be written as

$$\hat{y}_i = \sum_{j=0}^{i} \hat{G}_{i-j}(\hat{u}_j) \tag{5}$$

which is the convolution form of a system operating on the $L^p[0, \tau]$ valued signals $\{\hat{y}_i\}$ and $\{\hat{u}_i\}$. From (5) it is seen that $\{\hat{G}_i\}$ is the operator-valued impulse response of the lifting of G. From the above computation it is also seen that the convolution in (5) of the lifted signals and system is simply a rearrangement of the integral defining the operator. This rearrangement highlights the shift invariance property of the lifted system.

II. z-Transforms and Realizations of Lifted Systems

In the previous section we have seen that liftings of periodic continuous-time systems produced Banach space valued sequences representing signals and impulse responses. These sequences are discrete-time signals, so it is natural to try to develop a z-transform for these systems to exploit their shift-invariance property. This will also allow us to characterize the L^2 induced norm in the frequency domain as a type of an $\mathscr{H}^\infty$ norm [21, Chap. 5].

We begin abstractly with a sequence $\{h_i\}$ with values in a Banach space X, i.e., $\{h_i\} \in l_X$. The z-transform of $\{h_i\}$, written $H(z)$, is defined by

$$H(z) = \sum_{n=0}^{\infty} h_n z^n \tag{6}$$

for the values of the complex number z for which the series converges in X. Note that $H(z)$ is an X-valued function of a complex variable defined over some subset of the complex plane since for each z where (6) converges, $H(z) \in X$ by definition. By analogy with the classical z-transform, we expect the $H(z)$ to have some analytic type behavior in its region of convergence. Let $\Omega \subset \mathbb{C}$ be the domain of definition of this function, following [15], a notion of an X-valued analytic function can de defined. One of several equivalent definitions of analyticity is the following: we say that an X-valued function $H(z)$ defined on an open set $\Omega \subset \mathbb{C}$ is analytic in Ω if, for each $f \in X^*$, the complex valued function

$$\langle H(z), f \rangle$$

is analytic in Ω ($\langle \cdot, \cdot \rangle$ denotes the action of a linear functional in X^*, on an element in X). These analytic functions have many of the properties of the usual complex valued analytic functions including, for example, Cauchy's integral theorem and the maximum modulus principle.

If $\{h_i\} \in l_{L^p[0,\tau]}$, then the transform is an $L^p[0, \tau]$ valued function. If $\{\hat{G}_i\}$ is an impulse response sequence of a lifted system of the type considered in the previous section, so that $\{\hat{G}_i\}: l_{L^p[0,\tau]} \to l_{L^p[0,\tau]}$, then its transform, $\hat{G}(z)$ is an operator valued function, for each $z \in \Omega$, $\hat{G}(z) \in \mathscr{B}(L^p[0, \tau], L^p[0, \tau])$, the space of bounded operators on $L^p[0, \tau]$. To characterize the regions of convergence, we have the following theorem, which is an application of [15, Theorem 3.14.1].

Theorem 1: Let $\{h_i\} \in l_X$ and let $H(z)$ be its z-transform, then

i) If $\|h_i\|_X \le M < \infty \ \forall i$, then $H(z)$ is analytic in the region $\{|z| < 1\}$.

ii) If $\|h_i\|_X \le ka^i \ \forall i$, for some constants k and a, then $H(z)$ is analytic in the region $\{|z| < 1/a\}$.

This is a direct parallel to the case of the scalar valued z-transform with the absolute values being replaced by norms.

The same formal properties of the usual z-transform still hold in this general setting, for example, convolution in time is multiplication in frequency. To illustrate this, let $\{u_n\}, \{y_n\} \in l_{L^p[0,\tau]}$, and $\{G_n\}$ an operator sequence with $G_n \in \mathscr{B}(L^p[0, \tau]) \ \forall n$, such that

$$y_n = \sum_{m=0}^{n} G_{n-m} u_m.$$

Assume their z-transforms $U(z)$, $Y(z)$, and $G(z)$ are all convergent in some common region $\Delta_r = \{z; |z| < r\}$, then

$$Y(z) = \sum_{n=0}^{\infty} z^n y_n = \sum_{n=0}^{\infty} z^n \left(\sum_{m=0}^{n} G_{n-m} u_m \right)$$

which, by a change of variables and rearrangements of the

sum gives

$$Y(z) = \sum_{n=0}^{\infty} \sum_{k=n}^{0} z^{n+k} G_k u_{u-k} = \left(\sum_{i=0}^{\infty} z^i G_i \right) \left(\sum_{j=0}^{\infty} z^j u_j \right) = G(z)U(z) \quad \text{for } z \in \Delta_r.$$

The rearrangement of the sums above is permitted by the absolute convergence of all the series in the disk Δ_r. Note that the "multiplication" above is the operator $G(z)$ acting on $U(z)$. The other standard properties of the z-transform can be verified similarly.

We now look at the case of signals in L^2, we state the results from [21, Chap. 5] without proof, and summarize them in Theorem 2 below. Let $\hat{u} \in l^2_{L^2[0,\tau]}$, an $L^2[0, \tau]$-valued sequence whose norm sequence is square summable, i.e., $\sum_{n=0}^{\infty} \| \hat{u}_n \|^2_{L^2[0,\tau]} < \infty$. In the frequency domain, let $H(e^{i\theta})$, $0 \le \theta < 2\pi$, be an $L^2[0, \tau]$ valued function defined on the circle $T := \{e^{i\theta}; 0 \le \theta < 2\pi\}$. We define the frequency domain space:

$$L^2_{L^2[0,\tau]} = \left\{ H: T \to L^2[0, \tau]; \int_0^{2\pi} \| H(e^{i\theta}) \|^2_{L^2[0,\tau]} d\theta < \infty \right\}.$$

Note how this definition parallels that of the usual $L^2(T)$, but here the functions are Hilbert space valued and the integral is an integral of norms. Each $H \in L^2_{L^2[0,\tau]}$ has a Fourier series representation, i.e.,

$$H(e^{i\theta}) = \sum_{-\infty}^{\infty} h_n e^{in\theta}$$

where each $h_n \in L^2[0, \tau]$, and $\sum_{-\infty}^{\infty} \| h_n \|^2_{L^2[0,\tau]} < \infty$. An important subspace of $L^2_{L^2[0,\tau]}$ is $\mathscr{H}^2_{L^2[0,\tau]}$ which consists of those functions for which $h_n = 0$ for $n < 0$,

$$\mathscr{H}^2_{L^2[0,\tau]} = \left\{ H \in l^2_{L^2[0,\tau]}, \ H(e^{i\theta}) = \sum_{n=0}^{\infty} h_n e^{in\theta} \right\}.$$

Every function in $\mathscr{H}^2_{L^2[0,\tau]}$ can be extended to an analytic function inside the unit disk, and the space $\mathscr{H}^2_{L^2[0,\tau]}$ corresponds exactly to transforms of elements in $l^2_{L^2[0,\tau]}$.

To define operators on these spaces, consider a function $G(z)$ over the unit disk, which takes values in $\mathscr{B}(L^2[0, \tau])$ and has a power series representation

$$G(z) = \sum_{n=0}^{\infty} z^n G_n$$

where the coefficients $G_n \in \mathscr{B}(L^2[0, \tau])$. Let the series be convergent absolutely inside the disk, and suppose further that

$$\| G(z) \|_{\mathscr{B}(L^2[0,\tau])} \le M \qquad \forall |z| < 1.$$

Such a function will be called a *bounded analytic function* (on the unit disk). Such a function defines a bounded operator on $\mathscr{H}^2_{L^2[0,\tau]}$ by "multiplication," that is, for $Y, U \in \mathscr{H}^2_{L^2[0,\tau]}$, $Y = GU$ is defined by

$$Y(z) = G(z)U(z) \tag{7}$$

for each z. The definition makes sense since for each z, $G(z)$ is an operator on $L^2[0, \tau]$ and $U(z) \in L^2[0, \tau]$. If Y, U, and G are transforms of time-domain sequences, note that (7) defines an operation equivalent to convolution in the time domain.

Given G a bounded analytic function, we define a norm by

$$\| G \|_\infty = \sup_{|z|<1} \| G(z) \|_{\mathscr{B}(L^2[0,\tau])}.$$

We call the space of all bounded analytic functions $\mathscr{H}^\infty_{\mathscr{B}(L^2[0,\tau])}$ (or simply $\mathscr{H}^\infty$) over the unit disk, and again, the subscript $\mathscr{B}(L^2[0, \tau])$ denotes the space in which the function takes values. We call such functions operator valued since they take their values in a space of operators. $\mathscr{H}^\infty_{\mathscr{B}(L^2[0,\tau])}$ over the right-half plane is similarly defined as the space of all analytic operator valued functions over the right-half plane whose norm is uniformly bounded.

We now summarize with the following theorem.

Theorem 2:

i) The z-transform is an isometric isomorphism between the time domain space $l^2_{L^2[0,\tau]}$ and the frequency-domain space $\mathscr{H}^2_{L^2[0,\tau]}$.

ii) If G is a bounded analytic function, it defines a bounded operator on $\mathscr{H}^2_{L^2[0,\tau]}$ by the multiplication of (7), its induced norm is exactly $\| G \|_\infty$.

By the equivalence between a τ-periodic system and its lifting, this theorem provides a "frequency domain" characterization of the L^2 induced norm of a τ-periodic system. This characterization is not clear without the lifting. This also justifies calling the L^2-induced norm problem for sampled-data systems the $\mathscr{H}^\infty$ problem, since in the sequel we will be dealing with an equivalent lifted version of the sampled-data system.

We now consider state space realizations. A good state-space model to use for shift-invariant systems operating on $l_{L^p[0,\tau]}$ signals is the following:

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k + Du_k \tag{8}$$

with $u_k \in L^p[0, \tau]$, $y_k \in L^p[0, \tau]$, and $x_k \in X$ some Banach space (the state space). The system parameters $[A, B, C, D]$ are linear operators on the appropriate spaces, i.e.,

$$B: L^p[0, \tau] \to X$$
$$A: X \to X$$
$$C: X \to L^p[0, \tau]$$
$$D: L^p[0, \tau] \to L^p[0, \tau].$$

By simple finite algebraic operations, the system (8) can be represented by the convolution

$$y_k = \sum_{l=0}^{k} G_{k-l} u_l \tag{9}$$

where the impulse response $\{G_i\}$ is given by

$$\{G_i\} = \{D, CB, CAB, CA^2B, CA^3B, \cdots, \}. \tag{10}$$

The z-transform of the impulse response $\{G_i\}$ can be

computed from the realization using (10) as usual

$$G(z) = \sum_{n=0}^{\infty} z^n G_n = D + \sum_{n=1}^{\infty} z^n CA^{n-1}B$$
$$= D + Cz\left(\sum_{n=1}^{\infty} z^{n-1}A^{n-1}\right)B$$

and the series on the right converges in $\mathscr{B}(X)$ for $|z| < 1/\|A\|$, so

$$G(z) = D + Cz(I - zA)^{-1}B \qquad |z| < \frac{1}{\|A\|}. \quad (11)$$

In the important case when the state space is finite-dimensional, that is, $X = \mathbb{R}^N$, the situation is somewhat simplified. The transform in (11) defined for $\{|z| < 1/\|A\|\}$, can be analytically extended to the entire complex plane minus the finite set of reciprocals of the eigenvalues of the finite matrix A, i.e., in the region where $(I - zA)^{-1}$ is defined.

The usual rules of manipulation of realizations still hold in this more general setting, for example, composition, inversion, state transformation, etc. since they are based on formal manipulations.

III. Lifting the Hybrid System

In this section, we will show how the lifting technique can be used to convert the hybrid system to an equivalent system (in the sense of having equal induced norms), where the generalized plant is discrete-time time-invariant (though infinite-dimensional), and the controller is a discrete-time time-invariant system without any structural constraints.

First, we obtain state space realizations of lifted systems. An important fact here is that although finite-dimensional systems are lifted to systems with infinite-dimensional input and output spaces, the *state space* of the lifted systems will be shown to have at most the same dimension as that of the state space of the original systems, i.e., it is finite dimensional. Now, if $\hat{G}$ has a finite-dimensional realization, it would be of the following form:

$$\hat{G} = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right], \quad \text{with} \quad \begin{array}{l} \hat{B}: L^p[0,\tau] \to \mathbb{R}^x \\ \hat{A}: \mathbb{R}^x \to \mathbb{R}^x \\ \hat{C}: \mathbb{R}^x \to L^p[0,\tau] \\ \hat{D}: L^p[0,\tau] \to L^p[0,\tau] \end{array}$$

(The notation $\mathbb{R}^x, \mathbb{R}^u, \cdots$, stand for x being the dimension of the signal x and u the dimension of the signal u, etc.). The operator $\hat{B}: L^p[0,\tau] \to \mathbb{R}^x$ can be represented by a matrix of functions $\hat{B}(\hat{s})$, $\hat{B} \in [0,\tau]$, such that for a vector of functions $u \in L^p[0,\tau]$ we have

$$\hat{B}u = \int_0^\tau \hat{B}(\hat{s})u(\hat{s})\, d\hat{s}.$$

On the other hand, the operator $\hat{C}: \mathbb{R}^x \to L^p[0,\tau]$ (which is a finite rank operator, that is, it has a finite-dimensional range) is given by another matrix of functions $\hat{C}(\hat{t})$ $\hat{t} \in [0,\tau]$, such that for a vector $x \in \mathbb{R}^x$

$$\hat{C}x = \hat{C}(\hat{t})x, \qquad \hat{t} \in [0,\tau].$$

The class of operators $\hat{D}: L^p[0,\tau] \to L^p[0,\tau]$ that we will encounter have kernel representations, i.e., matrices of kernel functions $\hat{D}(\hat{t},\hat{s})$, such that for $u, y \in L^p[0,\tau]$, $y = \hat{D}u$ means

$$(\hat{D}u)(\hat{t}) = \int_0^\tau \hat{D}(\hat{t},\hat{s})u(\hat{s})\, d\hat{s}.$$

Notation: It simplifies the notation greatly to use the same symbol for an operator and its kernel for example, $\hat{D}(t,s)$ (or $\hat{B}(s)$) refer to the kernel functions representing the operator $\hat{D}$ (or $\hat{B}$), and $(\hat{D}^*\hat{D})(t,s)$ refers to the kernel function representing the operator $\hat{D}^*\hat{D}$. $\hat{A} = e^{A\tau}$ is the matrix representing the operator $\hat{A}: \mathbb{R}^n \to \mathbb{R}^n$. For operators that map a function space to $\mathbb{R}^n$, such as $\hat{B}$ above, we generally use s (or $\hat{s}$) to denote the variable of the kernel function, and for operators that map $\mathbb{R}^n$ to a function space such as $\hat{C}$ above, we use the variable t (or $\hat{t}$).

We now derive the operators $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ of a lifting in terms of the original system. Consider a finite-dimensional continuous time-invariant system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \qquad t \in (0,\infty). \end{aligned} \quad (12)$$

To obtain the lifting, we determine how the system relates the lifted signals $\{\hat{u}_k\}$ and $\{\hat{y}_k\}$. Let us define a new state which evolves in discrete time by $\hat{x}_k := x(k\tau)$. Assuming zero initial conditions, the behavior of the state in between samples, that is for $0 \le \hat{t} \le \tau$, is given by

$$\begin{aligned} x(k\tau + \hat{t}) &= \int_0^{k\tau+\hat{t}} e^{A(k\tau+\hat{t}-s)}Bu(s)\, ds \\ &= \int_0^{k\tau} e^{A(k\tau+\hat{t}-s)}Bu(s)\, ds \\ &\quad + \int_{k\tau}^{k\tau+\hat{t}} e^{A(k\tau+\hat{t}-s)}Bu(s)\, ds \\ &= e^{A\hat{t}}\int_0^{k\tau} e^{A(k\tau-s)}Bu(s)\, ds \\ &\quad + \int_0^{\hat{t}} e^{A(k\tau+\hat{t}-(k\tau+\hat{s}))}Bu(k\tau+\hat{s})\, d\hat{s} \\ &= e^{A\hat{t}}x(k\tau) + \int_0^{\hat{t}} e^{A(\hat{t}-\hat{s})}B\,\hat{u}_k(\hat{s})\, d\hat{s}. \end{aligned} \quad (13)$$

In particular, the new state $\hat{x}$ evolves by

$$\hat{x}_{k+1} = e^{A\tau}\hat{x}_k + \int_0^\tau e^{A(\tau-\hat{s})}B\,\hat{u}_k(\hat{s})\, d\hat{s} \quad (14)$$

or, in operator notation

$$\hat{x}_{k+1} = e^{A\tau}\hat{x}_k + \hat{B}\,\hat{u}_k$$

where $\hat{B}$ is the $L^p[0,\tau] \to \mathbb{R}^x$ operator defined by the kernel $e^{A(\hat{t}-\hat{s})}B$. As for the output signal, $\{\hat{y}_k\}$ can be

written as

$$\begin{aligned}\hat{y}_k(\hat{t}) &= y(k\tau + \hat{t}) = Cx(k\tau + \hat{t}) + Du(k\tau + \hat{t})\\ &= C\left[e^{A\hat{t}}x(k\tau) + \int_0^{\hat{t}} e^{A(\hat{t}-\hat{s})}B\hat{u}_k(\hat{s})\,d\hat{s}\right] + D\hat{u}_k(\hat{t})\\ &= Ce^{A\hat{t}}\hat{x}_k + \int_0^{\tau}\left[Ce^{A(\hat{t}-\hat{s})}\mathbf{1}_{(\hat{t}-\hat{s})}B + D\delta(\hat{t}-\hat{s})\right]\\ &\quad\cdot \hat{u}_k(\hat{s})\,d\hat{s} \qquad (15)\end{aligned}$$

with $0 \le \hat{t} \le \tau$, and where $\mathbf{1}_{(t)}$ is the unit step function

$$\mathbf{1}_{(t)} = \begin{cases} 1 & t > 0\\ 0 & t \le 0.\end{cases}$$

Equation (15) in operator notation is

$$\hat{y}_k = \hat{C}\hat{x}_k + \hat{D}\hat{u}_k$$

where $\hat{C}$ and $\hat{D}$ are given by the kernels in (15).

In summary, a time-invariant system G given by (12) has a lifting given by

$$\hat{G} = \left[\begin{array}{c|c} \hat{A} & \hat{B}\\ \hline \hat{C} & \hat{D}\end{array}\right] = \left[\begin{array}{c|c} e^{A\tau} & e^{A(\tau-\hat{s})}B\\ \hline Ce^{A\hat{t}} & Ce^{A(\hat{t}-\hat{s})}\mathbf{1}_{(\hat{t}-\hat{s})}B + D\delta(\hat{t}-\hat{s})\end{array}\right]. \qquad (16)$$

Note that the operator $\hat{D}$ is the restriction of the original system G to the input subspace $L^2[0,\tau]$, that is $\hat{D} = \Pi_{L^2[0,\tau]}G\,|_{L^2[0,\tau]}$.

The important conclusion to be made here is that the state space of the lifted systems can be chosen to be finite dimensional. This is in contrast to [26], where a similar lifting technique was developed, but the state space of the lifted systems is infinite-dimensional. As we will see in the next section, in the solution of the $\mathscr{H}^\infty$ sampled-data problem it is of primary importance (in fact, it is what makes the solution possible) that the state space of the lifted systems be finite dimensional.

We now consider a time-invariant system with a sampler on the measurement output and a hold on the control input as shown in Fig. 4. The filter F is some strictly proper system, this is required for the sampling operation to be well defined. We can absorb F into the description of G and simply assume that we are given G with a realization

$$G = \begin{bmatrix} G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2\\ \hline C_1 & D_{11} & D_{12}\\ C_2 & 0 & 0\end{array}\right]$$

where $D_{21} = D_{22} = 0$ because F is strictly proper. This guarantees that the measurement outputs are continuous functions of time.

The sampler produces the discrete-time signal $\tilde{y}$ from the continuous-time signal y by sampling it at times $k\tau$, and the hold produces the piecewise constant continuous-time signal u from the discrete-time signal $\tilde{u}$. It is helpful to view the sampler and hold as relating the discrete-time signals $\{\tilde{u}_k\}, \{\tilde{y}_k\}$ to the $L^p[0,\tau]$-valued discrete-time signals $\{\hat{u}_k\}$ and $\{\hat{y}_k\}$, the liftings of u and y

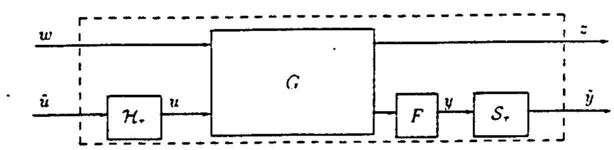

Fig. 4. Plant with sample and a hold.

$$\hat{\mathscr{H}}_\tau: \mathbb{R}^u \to L^p[0,\tau];\ \hat{u}_k = \hat{\mathscr{H}}_\tau\tilde{u}_k \Leftrightarrow \hat{u}_k(\hat{t}) = \tilde{u}_k \qquad 0 \le \hat{t} \le \tau$$

$$\hat{\mathscr{S}}_\tau: L^p[0,\tau] \to \mathbb{R}^y;\ \tilde{y}_k = \hat{\mathscr{S}}_\tau\hat{y}_k \Leftrightarrow \tilde{y}_k = \hat{y}_k(0).$$

Note that $\hat{\mathscr{S}}_\tau$ is not well defined on $L^p[0,\tau]$, but on the subspace of continuous functions in $L^p[0,\tau]$, this distinction will be irrelevant since in our use of $\hat{\mathscr{S}}_\tau$, assumption are made (i.e., the presence of the strictly causal filter F above) to guarantee that $\hat{\mathscr{S}}_\tau$ only operates on continuous signals. Specifically, $\hat{\mathscr{S}}_\tau$ is only used in expressions like $\hat{\mathscr{S}}_\tau T$, where T will always be an operator whose range is made up of continuous functions.

The lifting $\hat{G}$ given by

$$\hat{G} = \begin{bmatrix} \hat{G}_{11} & \hat{G}_{12}\\ \hat{G}_{21} & \hat{G}_{22}\end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2\\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12}\\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22}\end{array}\right]$$

relates the signals $\hat{w}, \hat{z}, \hat{u}, \hat{y}$. On the other hand, the system $\tilde{G}$ (see Fig. 5(b)) given by

$$\tilde{G} = \begin{bmatrix} \hat{G}_{11} & \hat{G}_{12}\hat{\mathscr{H}}_\tau\\ \hat{\mathscr{S}}_\tau\hat{G}_{21} & \hat{\mathscr{S}}_\tau\hat{G}_{22}\hat{\mathscr{H}}_\tau\end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2\hat{\mathscr{H}}_\tau\\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12}\hat{\mathscr{H}}_\tau\\ \hat{\mathscr{S}}_\tau\hat{C}_2 & \hat{\mathscr{S}}_\tau\hat{D}_{21} & \hat{\mathscr{S}}_\tau\hat{D}_{22}\hat{\mathscr{H}}_\tau\end{array}\right];$$

$$\begin{bmatrix}\hat{z}\\ \tilde{y}\end{bmatrix} = \tilde{G}\begin{bmatrix}\hat{w}\\ \tilde{u}\end{bmatrix} \qquad (17)$$

relates the signals $\hat{w}, \hat{z}, \tilde{y}, \tilde{u}$. This formulation shows one of the advantages of the lifting, in that all signals in the system are viewed over the same time set (discrete time) without losing any part of the continuous-time signals w and z. The signals $\tilde{u}, \tilde{y}$ take values in $\mathbb{R}^u$ and $\mathbb{R}^y$ but $\hat{w}$ and $\hat{z}$ take values in the much larger space $L^p[0,\tau]$.

We now explicitly evaluate the operators in (17). $\hat{B}_2\hat{\mathscr{H}}_\tau$ is a matrix obtained by feeding $\hat{B}_2$ a constant input, i.e.,

$$\hat{B}_2\hat{\mathscr{H}}_\tau = \int_0^\tau e^{A(\tau-s)}B_2\,ds = \left(\int_0^\tau e^{Ar}dr\right)B_2 = \Psi(\tau)B_2$$

(where $\Psi(t) := \int_0^t e^{Ar}\,dr$). $\hat{\mathscr{S}}_\tau\hat{C}_2$ is obtained by

$$\int_0^\tau \delta(t)C_2e^{At}\,dt = C_2.$$

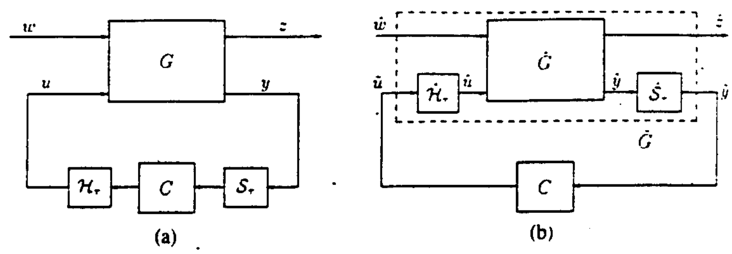

Fig. 5. (a) Hybrid system. (b) Lifted system with discrete time controller.

Similarly

$$\begin{aligned}\hat{D}_{12}\hat{\mathscr{H}}_\tau &= \int_0^\tau \left[C_1 e^{A(t-s)}1_{(t-s)}B_2 + D_{12}\delta(t-s)\right] ds\\ &= C_1\left(\int_0^t e^{Ar}dr\right)B_2 + D_{12}\\ &= C_1\Psi(t)B_2 + D_{12}.\end{aligned}$$

Note that $\hat{D}_{12}\hat{\mathscr{H}}_\tau$ is given by its kernel, a function of a single variable t, since $\hat{D}_{12}\hat{\mathscr{H}}_\tau$ is an operator from $\mathbb{R}^u \to L^p[0, \tau]$. We also compute

$$\hat{\mathscr{S}}_\tau \hat{D}_{21} = \int_0^\tau \delta(t) C_2 e^{A(t-s)}1_{(t-s)}B_1\, ds = 0$$

and similarly, $\hat{\mathscr{S}}_\tau \hat{D}_{22} = 0$ implying that $\hat{\mathscr{S}}_\tau \hat{D}_{22}\hat{\mathscr{H}}_\tau = 0$.

In summary, we arrive at the following realization for $\tilde{G}$

$$\begin{aligned}\tilde{G} = \begin{bmatrix}\tilde{G}_{11} & \tilde{G}_{12}\\ \tilde{G}_{21} & \tilde{G}_{22}\end{bmatrix} &= \left[\begin{array}{c|cc}\hat{A} & \hat{B}_1 & \tilde{B}_2\\ \hline \hat{C}_1 & \hat{D}_{11} & \tilde{D}_{12}\\ \tilde{C}_2 & 0 & 0\end{array}\right]\\ &= \left[\begin{array}{c|cc} e^{A\tau} & e^{A(\tau-s)}B_1 & \Psi(\tau)B_2\\ \hline C_1 e^{At} & \hat{D}_{11} & C_1\Psi(t)B_2 + D_{12}\\ C_2 & 0 & 0\end{array}\right]. \quad (18)\end{aligned}$$

Note that the four subsystems in $\tilde{G}$ are shift-invariant operators with the following input–output spaces:

$$\begin{aligned}&\tilde{G}_{11}: l_{L^p[0,\tau]} \to l_{L^p[0,\tau]}\\ &\tilde{G}_{12}: l_{\mathbb{R}^u} \to l_{L^p[0,\tau]}\\ &\tilde{G}_{21}: l_{L^p[0,\tau]} \to l_{\mathbb{R}^y}\\ &\tilde{G}_{22}: l_{\mathbb{R}^u} \to l_{\mathbb{R}^y}.\end{aligned}$$

We now comment on the synthesis problem for the hybrid system using this new setting. Let us adopt the notation $\mathscr{F}(P, K)$ as referring to a generalized plant P in feedback with K, and also to the resulting closed-loop mapping between the exogeneous input and the regulated output. Fig. 5(a) shows the original hybrid system $\mathscr{F}(G, \mathscr{H}_\tau C\mathscr{S}_\tau)$ and Fig. 5(b) shows the lifted system $\tilde{G}$ with the sampler, hold and controller $\mathscr{F}(\tilde{G}, C)$.

Since $\hat{w} = W_\tau w$ and $\hat{z} = W_\tau z$, then the closed-loop systems are related by

$$\mathscr{F}(\tilde{G}, C) = W_\tau \mathscr{F}(G, \mathscr{H}_\tau C S_\tau) W_\tau^{-1}$$

or in other words, the closed-loop operator $\mathscr{F}(\tilde{G}, C)$ is the *lifting* of the closed-loop operator of the hybrid system $\mathscr{F}(G, \mathscr{H}_\tau C\mathscr{S}_\tau)$. By the isometry properties of the lifting W_τ, we have that

$$\|\mathscr{F}(G, \mathscr{H}_\tau C\mathscr{S}_\tau)\| = \|\mathscr{F}(\tilde{G}, C)\| \quad (19)$$

where the norms are the L^p-induced norm on $\mathscr{F}(G, \mathscr{H}_\tau C\mathscr{S}_\tau)$, and the $l^p_{L^p[0,\tau]}$-induced norm on $\mathscr{F}(\tilde{G}, C)$, and note that the same controller C is on both sides of the equation in (19).

The consequence of (19) is that the design problem for the continuous-time plant G with a sampled-data controller $\mathscr{H}_\tau C\mathscr{S}_\tau$, is equivalent to the design problem for the generalized plant $\tilde{G}$ and the controller C. The advantage of this reformulation is three fold; first, both the generalized plant and the controller are discrete-time (thus the hybrid nature of the system is no longer problematic); second, both $\tilde{G}$ and C are time-invariant (thus "removing" the periodicity from the system); third, the controller C has no structural constraints on it (i.e., that it be a sampled-data controller of the form $\mathscr{H}_\tau C\mathscr{S}_\tau$). The price paid for these advantages is the infinite-dimensionality of the exogeneous input and regulated output spaces. However, as will be seen in the next section, the infinite-dimensionality of the input and output spaces can essentially be "removed" without affecting the norm.

The equivalence just mentioned is not complete without addressing the issue of the stability of the hybrid system. By internal stability, we mean that the state of the system is exponentially convergent to zero given any initial conditions. The following theorem provides an equivalence between the stability of the hybrid system and the stability of a discrete-time system, it is essentially a restatement of [11, Theorem 4].

Theorem 3: A controller C internally stabilizes the hybrid system in Fig. 5(a) if and only if it internally stabilizes the discrete-time system $\mathscr{S}_\tau \hat{G}_{22}\mathscr{H}_\tau$ (Fig. 6).

Note that since $\mathscr{S}_\tau \hat{G}_{22}\mathscr{H}_\tau$ is a finite-dimensional discrete-time system, the stability of the hybrid system is well understood. In particular, one can parametrize all (finite dimensional) controllers that stabilize the hybrid system as the (finite-dimensional) controllers that stabilize $\mathscr{S}_\tau \hat{G}_{22}\mathscr{H}_\tau$.

Finally, we comment that, as is well known $\mathscr{S}_\tau \hat{G}_{22}\mathscr{H}_\tau$ may not be stabilizable even if G_{22} is, i.e., even if (A, B_2) is stabilizable, there is a discrete set of sampling periods $\{\tau_i\}$ such that $(e^{A\tau_i}, \Psi(\tau_i)B_2)$ might lose controllability of certain eigenvalues. But if τ is chosen outside of the set where $(C_2, e^{A\tau})$ is not detectable and $(e^{A\tau}, \Psi(\tau)B)$ is not stabilizable, then we guarantee that $\mathscr{S}_\tau \hat{G}_{22}\mathscr{H}_\tau$ (and thus the hybrid system) is stabilizable if (C_2, A) is detectable and (A, B_2) is

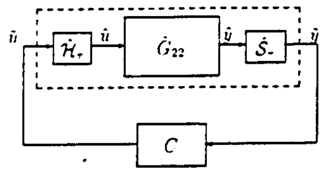

Fig. 6. C in feedback with the discretized system $\mathcal{S}_\tau \hat{G}_{22} \mathcal{H}_\tau$.

stabilizable, we call such sampling periods *nonpathological* and for the remainder of the paper we assume that τ is nonpathological.

IV. The $\mathcal{H}^\infty$ Problem for Hybrid Systems

We now apply the lifting technique to solve the $\mathcal{H}^\infty$ problem for the hybrid system of Fig. 1. Specifically, we will find all controllers C (if they exist) such that the closed-loop L^2 induced norm is less than some prespecified level, i.e., $\|\mathcal{F}(G, \mathcal{H}_\tau C \mathcal{S}_\tau)\| < \gamma$. This is done by establishing an equivalence between the hybrid system and a certain finite-dimensional discrete-time system, in the sense that the L^2 induced norm of the hybrid system is less than a prespecified γ if and only if the $\mathcal{H}^\infty$ norm of a discrete-time system is less than γ. The latter problem is well understood in the literature, and its solution provides all controllers that constrain the closed-loop norm to be less than γ. A basic fact that we use in our constructions is that the $\mathcal{H}^\infty$ norm is the induced operator norm on a *Hilbert space*. This allows us to apply the geometric structure of the underlying Hilbert space.

Consider Fig. 5. By the isometry properties of the lifting and the stability discussion in the previous section, the following two statements are equivalent

i) $\mathcal{H}_\tau C \mathcal{S}_\tau$ internally stabilizes G and $\|\mathcal{F}(G, \mathcal{H}_\tau C \mathcal{S}_\tau)\| < \gamma$;

ii) C internally stabilizes $\tilde{G}$ and $\|\mathcal{F}(\tilde{G}, C)\| < \gamma$.

The induced norms are over $L^2[0, \infty)$ and $l^2_{L^2[0,\tau]}$, respectively. Therefore, from now on we will be concerned with the second problem involving $\tilde{G}$ (note that to simplify notation γ will be considered 1 from now on, the general case follows as usual by scaling).

Our approach will be to establish a further equivalence between ii) and a finite-dimensional discrete-time problem. This is done in two steps, the first is obtain from $\tilde{G}$ another system $\overline{G}$ with

$$\overline{G} = \left[\begin{array}{c|cc} \overline{A} & \overline{B}_1 & \overline{B}_2 \\ \hline \overline{C}_1 & 0 & \overline{D}_{12} \\ \overline{C}_2 & 0 & 0 \end{array}\right] \tag{20}$$

such that $\|\mathcal{F}(\tilde{G}, C)\| < 1$ if and only if $\|\mathcal{F}(\overline{G}, C)\| < 1$. The second step is to reduce the problem with $\overline{G}$ to a finite-dimensional problem.

We describe the second step first. Given $\overline{G}$ with a realization as above, the operators in the realization have the same input–output spaces as the corresponding operators in $\tilde{G}$, namely

$$\overline{G} = \left[\begin{array}{c|cc} \overline{A} & \overline{B}_1 & \overline{B}_2 \\ \hline \overline{C}_1 & 0 & \overline{D}_{12} \\ \overline{C}_2 & 0 & 0 \end{array}\right];$$

$$\begin{array}{ll} \overline{A}: \mathbb{R}^x \to \mathbb{R}^x & \overline{B}_1: L^2[0, \tau] \to \mathbb{R}^x \\ \overline{B}_2: \mathbb{R}^u \to \mathbb{R}^x & \overline{C}_1: \mathbb{R}^x \to L^2[0, \tau] \\ \overline{C}_2: \mathbb{R}^x \to \mathbb{R}^y & \overline{D}_{12}: \mathbb{R}^u \to L^2[0, \tau] \end{array}. \tag{21}$$

We are interested in characterizing all controllers C such that $\mathcal{F}(\overline{G}, C)$ is internally stable and $\|\mathcal{F}(\overline{G}, C)\| < 1$. The basic idea is that since the state space is finite-dimensional, then the infinite-dimensional operators $\overline{B}_1, \overline{C}_1, \overline{D}_{12}$ are actually finite rank operators, and by examining their range and null spaces closely, the problem can be reduced to a finite dimensional one.

Let us denote by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ the null and range spaces of a given operator T, respectively. Consider first the operator $\overline{B}_1: L^2[0, \tau] \to \mathbb{R}^x$, its initial space can be decomposed as $L^2[0, \tau] = \mathcal{N}(\overline{B}_1) \oplus \mathcal{N}(\overline{B}_1)^\perp$; where $\mathcal{N}(\overline{B}_1)^\perp := L^2[0, \tau) \ominus \mathcal{N}(\overline{B}_1)$. With respect to this decomposition, $\overline{B}_1$ has the following "block matrix" representation

$$\overline{B}_1 = \begin{bmatrix} 0 & \breve{B}_1 \end{bmatrix}: \quad \begin{array}{c} \mathcal{N}(\overline{B}_1) \\ \oplus \\ \mathcal{N}(\overline{B}_1)^\perp \end{array} \to \mathbb{R}^x.$$

An important point here is that since $\overline{B}_1$ has a finite-dimensional range, then $\overline{B}_1$ is zero on all but a finite-dimensional subspace of $L^2[0, \tau]$, that is, $\mathcal{N}(\overline{B}_1)^\perp$ is finite-dimensional. The nonzero part of $\overline{B}_1$, namely $\breve{B}_1 := \overline{B}_1|_{\mathcal{N}(\overline{B}_1)^\perp}$, is a linear mapping between finite-dimensional Hilbert spaces. The decomposition of the operator $\overline{B}_1$ induces a decomposition on the input signal $\hat{w}$, by

$$\hat{w} = \begin{bmatrix} \hat{w}_i \\ \hat{w}_f \end{bmatrix},$$

$\hat{w}_i \in l_{\mathcal{N}(\overline{B}_1)}$, and $\hat{w}_f \in l_{\mathcal{N}(\overline{B}_1)^\perp}$. Note that $\hat{w}_i$ is an infinite-dimensional signal while $\hat{w}_f$ is a finite-dimensional signal.

A similar argument works for decomposing the output space and the signal $\hat{z}$. Define

$$\mathcal{R}(\overline{C}_1, \overline{D}_{12}) := \mathcal{R}(\overline{C}_1) + \mathcal{R}(\overline{D}_{12})$$

and note that $\mathcal{R}(\overline{C}_1, \overline{D}_{12})$ is finite-dimensional since both $\overline{C}_1$ and $\overline{D}_{12}$ have finite-dimensional ranges. We now decompose the output space and the $\overline{C}_1, \overline{D}_{12}$ operators as follows:

$$\overline{C}_1 = \begin{bmatrix} 0 \\ \breve{C}_1 \end{bmatrix}: \quad \mathbb{R}^x \to \begin{array}{c} \mathcal{R}(\overline{C}_1, \overline{D}_{12})^\perp \\ \oplus \\ \mathcal{R}(\overline{C}_1, \overline{D}_{12}) \end{array};$$

$$\overline{D}_{12} = \begin{bmatrix} 0 \\ \breve{D}_{12} \end{bmatrix}: \quad \mathbb{R}^u \to \begin{array}{c} \mathcal{R}(\overline{C}_1, \overline{D}_{12})^\perp \\ \oplus \\ \mathcal{R}(\overline{C}_1, \overline{D}_{12}). \end{array}$$

And similarly, the output signal $\hat{z}$ can be decomposed into

$$\hat{z} = \begin{bmatrix} \hat{z}_i \\ \hat{z}_f \end{bmatrix}; \quad \hat{z}_i \in l_{\mathcal{R}(\overline{C}_1, \overline{D}_{12})^\perp}, \quad \hat{z}_f \in l_{\mathcal{R}(\overline{C}_1, \overline{D}_{12})}.$$

Note that $\hat{z}_i$ and $\hat{z}_f$ are infinite- and finite-dimensional

signals, respectively. The decompositions make $\bar{G}$ into a three-input three-output system given by

$$\bar{G} = \left[\begin{array}{c|ccc} \bar{A} & 0 & \breve{B}_1 & \bar{B}_2 \\ \hline 0 & 0 & 0 & 0 \\ \breve{C}_1 & 0 & 0 & \breve{D}_{12} \\ \bar{C}_2 & 0 & 0 & 0 \end{array}\right]: \begin{array}{c} l_{\mathcal{N}(\bar{B}_1)} \\ \oplus \\ l_{\mathcal{N}(\bar{B}_1)^\perp} \\ \oplus \\ l_{\mathbb{R}^u} \end{array} \rightarrow \begin{array}{c} l_{\mathcal{R}(\bar{C}_1, \bar{D}_{12})^\perp} \\ \oplus \\ l_{\mathcal{R}(\bar{C}_1, \bar{D}_{12})} \\ \oplus \\ l_{\mathbb{R}^y} \end{array}.$$

Define the subsystem

$$\breve{G} = \left[\begin{array}{c|cc} \bar{A} & \breve{B}_1 & \bar{B}_2 \\ \hline \breve{C}_1 & 0 & \breve{D}_{12} \\ \bar{C}_{12} & 0 & 0 \end{array}\right]: \begin{array}{c} l_{\mathcal{N}(B_1)^\perp} \\ \oplus \\ l_{\mathbb{R}^u} \end{array} \rightarrow \begin{array}{c} l_{\mathcal{R}(\bar{C}_1, \bar{D}_{12})} \\ \oplus \\ l_{\mathbb{R}^y} \end{array}. \tag{22}$$

$\bar{G}$ can now be rewritten in terms of $\breve{G}$ as

$$\bar{G} = \begin{bmatrix} \breve{G} & 0 \\ 0 & 0 \end{bmatrix}: \begin{array}{c} l_{\mathcal{N}(\bar{B}_1)^\perp \oplus \mathbb{R}^u} \\ \oplus \\ l_{\mathcal{N}(\bar{B}_1)} \end{array} \rightarrow \begin{array}{c} l_{\mathcal{R}(\bar{C}_1, \bar{D}_{12}) \oplus \mathbb{R}^y} \\ \oplus \\ l_{\mathcal{R}(\bar{C}_1, \bar{D}_{12})^\perp} \end{array}.$$

It now follows that if the controller $u = Cy$ is connected to both $\bar{G}$ and $\breve{G}$ (Fig. 7) then

$$\mathcal{F}(\bar{G}, C) = \begin{bmatrix} \mathcal{F}(\breve{G}, C) & 0 \\ 0 & 0 \end{bmatrix}. \tag{23}$$

We thus conclude that $\mathcal{F}(\bar{G}, C)$ is internally stable and $\|\mathcal{F}(\bar{G}, C)\| < 1$ if and only if $\mathcal{F}(\breve{G}, C)$ is internally stable and $\|\mathcal{F}(\breve{G}, C)\| < 1$. The equivalence of the norm bounds follows trivially from (23), and the equivalence of internal stability follows from the fact that $\mathcal{F}(\bar{G}, C)$ and $\mathcal{F}(\breve{G}, C)$ have the same "A" matrices.

The new input and output signals $\hat{z}_f$ and $\hat{w}_f$ take values in the finite-dimensional Hilbert spaces $\mathcal{R}(\bar{C}_1, \bar{D}_{12})$ and $\mathcal{N}(\bar{B}_1)^\perp$, respectively. Any finite-dimensional Hilbert space of dimension n is isometrically isomorphic to the Euclidean space $\mathbb{R}^n$, thus with the proper identification of $\mathcal{R}(\bar{C}_1, \bar{D}_{12})$ and $\mathcal{N}(\bar{B}_1)^\perp$ with Euclidean spaces, the problem with $\breve{G}$ is reduced to a standard finite-dimensional discrete-time $\mathcal{H}^\infty$ problem. This is done in the next theorem.

Theorem 4: Given the infinite-dimensional system $\bar{G}$ defined by (21), form the $b \times b$ nonsingular diagonal matrix Σ_b and the $cd \times cd$ nonsingular diagonal matrix Σ_{cd} from the following symmetric factorizations

$$\bar{B}_1 \bar{B}_1^* = T_B^* \begin{bmatrix} \Sigma_b & 0 \\ 0 & 0 \end{bmatrix} T_B; \quad \begin{bmatrix} \bar{C}_1^* \\ \bar{D}_{12} \end{bmatrix} \begin{bmatrix} \bar{C}_1 & \bar{D}_{12} \end{bmatrix}$$

$$= T_{CD}^* \begin{bmatrix} \Sigma_{cd} & 0 \\ 0 & 0 \end{bmatrix} T_{CD}.$$

Define the finite-dimensional system

$$\acute{G} = \left[\begin{array}{c|cc} \bar{A} & \acute{B}_1 & \bar{B}_2 \\ \hline \acute{C}_1 & 0 & \acute{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right]: \begin{array}{c} l_{\mathbb{R}^b} \\ \oplus \\ l_{\mathbb{R}^u} \end{array} \rightarrow \begin{array}{c} l_{\mathbb{R}^{cd}} \\ \oplus \\ l_{\mathbb{R}^y} \end{array}$$

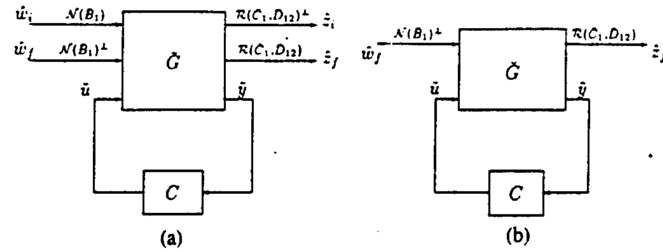

Fig. 7. (a) $\mathcal{F}(\bar{G}, C)$. (b) $\mathcal{F}(\breve{G}, C)$.

where

$$\acute{B}_1 := T_B^* \begin{bmatrix} \Sigma_b^{1/2} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \acute{C}_1 & \acute{D}_{12} \end{bmatrix} := \begin{bmatrix} \Sigma_{cd}^{1/2} & 0 \end{bmatrix} T_{CD}.$$

Then

i) $b := \operatorname{rank}\{\bar{B}_1 \bar{B}_1^*\} = \dim\{\mathcal{N}(\bar{B}_1)^\perp\}$.

$$cd := \operatorname{rank}\left\{\begin{bmatrix} \bar{C}_1^* \\ \bar{D}_{12}^* \end{bmatrix} [\bar{C}_1 \bar{D}_{12}]\right\} = \dim\{\mathcal{R}(\bar{C}_1, \bar{D}_{12})\};$$

ii) For a discrete-time time-invariant controller C

a) $\mathcal{F}(\acute{G}, C)$ is internally stable if and only if $\mathcal{F}(\bar{G}, C)$ is;

b) $\|\mathcal{F}(\acute{G}, C)\| = \|\mathcal{F}(\bar{G}, C)\|$.

Remark: $\acute{G}$ here is simply $\breve{G}$ defined previously but with a particular choice of orthonormal basis for $\mathcal{N}(\bar{B}_1)^\perp$ and $\mathcal{R}(\bar{C}_1, \bar{D}_{12})$, and the matrices $\acute{B}_1, \acute{C}_1, \acute{D}_{12}$ represent the operators $\breve{B}_1, \breve{C}_1, \breve{D}_{12}$ in this particular basis.

Proof: To show i) note that for any Hilbert space operator T

$$\dim\{\mathcal{R}(T)\} = \dim\{\mathcal{N}(T)^\perp\};$$
$$\mathcal{R}(T) = \mathcal{R}(TT^*) = \mathcal{R}(TT^*T). \tag{24}$$

The first identity is standard since for any linear operator $T: H_1 \rightarrow H_2$, we have $H_1/\mathcal{N}(T) \simeq \mathcal{R}(T)$, and in a Hilbert space $H_1/\mathcal{N}(T) \simeq \mathcal{N}(T)^\perp$. The first equality in the second identity follows from $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$, and the second equality follows from $\mathcal{R}(T) = \mathcal{N}(T^*)^\perp$.

Now

$$\operatorname{rank}\{\bar{B}_1 \bar{B}_1^*\} := \dim\{\mathcal{R}(\bar{B}_1 \bar{B}_1^*)\} = \dim\{\mathcal{R}(\bar{B}_1)\}$$
$$= \dim\{\mathcal{N}(\bar{B}_1)^\perp\}.$$

For $\mathcal{R}(\bar{C}_1 \bar{D}_{12})$, note that $\mathcal{R}(\bar{C}_1, \bar{D}_{12}) = \mathcal{R}([\bar{C}_1 \quad \bar{D}_{12}])$, and

$$\operatorname{rank}\left\{\begin{bmatrix} \bar{C}_1^* \\ \bar{D}_{12}^* \end{bmatrix} [\bar{C}_1 \quad \bar{D}_{12}]\right\}$$

$$:= \dim\left\{\mathcal{R}\left(\begin{bmatrix} \bar{C}_1^* \\ \bar{D}_{12}^* \end{bmatrix} [\bar{C}_1 \quad \bar{D}_{12}]\right)\right\}$$

$$= \dim\left\{\mathcal{R}\left(\begin{bmatrix} \bar{C}_1^* \\ \bar{D}_{12}^* \end{bmatrix}\right)\right\}$$

$$= \dim\left\{\mathcal{N}([\bar{C}_1 \quad \bar{D}_{12}])^\perp\right\} = \dim\{\mathcal{R}([\bar{C}_1 \quad \bar{D}_{12}])\}.$$

To show ii) b), recall from the earlier discussion that the

internal stability and norms of $\mathscr{F}(\bar{G}, C)$ and $\mathscr{F}(\check{G}, C)$ are equivalent. $\check{G}$ was given by

$$\check{G} = \left[\begin{array}{c|cc} \bar{A} & \check{B}_1 & \bar{B}_2 \\ \hline \check{C}_1 & 0 & \check{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right] : \begin{array}{c} l_{\mathscr{N}(\bar{B}_1)^\perp} \\ \oplus \\ l_{\mathbb{R}^u} \end{array} \rightarrow \begin{array}{c} l_{\mathscr{R}(\bar{C}_1, \bar{D}_{12})} \\ \oplus \\ l_{\mathbb{R}^y} \end{array}$$

with $\check{B}_1 := \bar{B}_1 \rfloor_{\mathscr{N}(\bar{B}_1)^\perp}$; $[\check{C}_1 \ \check{D}_{12}] := \Pi_{\mathscr{R}(\bar{C}_1, \bar{D}_{12})}[\bar{C}_1 \ \bar{D}_{12}]$. From i), $\mathscr{N}(\bar{B}_i)^\perp$ and $\mathscr{R}(\bar{C}_1, \bar{D}_{12})$ are isometrically isomorphic to $\mathbb{R}^b$ and $\mathbb{R}^{cd}$, respectively. To find this isomorphism, note that

$$\begin{aligned} \mathscr{N}(\bar{B}_1)^\perp &= \mathscr{R}(\bar{B}_1^*) = \mathscr{R}(\bar{B}_1^* \bar{B}_1 \bar{B}_1^*) \\ &= \mathscr{R}\left(\bar{B}_1^* T_B^* \begin{bmatrix} \Sigma_b & 0 \\ 0 & 0 \end{bmatrix} T_B\right) \\ &= \mathscr{R}\left(\bar{B}_1^* T_B^* \begin{bmatrix} \Sigma_b & 0 \\ 0 & 0 \end{bmatrix}\right) \end{aligned}$$

where the last equality follows because $\mathscr{R}(TY) = \mathscr{R}(T)$ whenever Y is invertible. Furthermore

$$\begin{aligned} \mathscr{N}(\bar{B}_1)^\perp &= \mathscr{R}\left(\bar{B}_1^* T_B^* \begin{bmatrix} \Sigma_b & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathscr{R}\left(\bar{B}_1^* T_B^* \begin{bmatrix} \Sigma_b \\ 0 \end{bmatrix}\right) \\ &= \mathscr{R}\left(\bar{B}_1^* T_B^* \begin{bmatrix} \Sigma_b^{-1/2} \\ 0 \end{bmatrix}\right). \end{aligned} \tag{25}$$

So define an operator

$$U: \mathbb{R}^b \rightarrow \mathscr{N}(\bar{B}_1)^\perp \subset L^2[0, \tau] \text{ by } U := \bar{B}_1^* T_B^* \begin{bmatrix} \Sigma_b^{-1/2} \\ 0 \end{bmatrix}.$$

U is an isometry (an isometry U is an operator that preserves inner products, i.e., $\langle Ux, Uy \rangle = \langle x, y \rangle \ \forall x, y$, this is equivalent to $U^*U = I$), and from (25) it is onto $\mathscr{N}(\bar{B}_1)^\perp$, thus it is an isometric isomorphism between $\mathbb{R}^b$ and $\mathscr{N}(\bar{B}_1)^\perp$. Define $\acute{B}_1: \mathbb{R}^b \rightarrow \mathbb{R}^x$ by $\acute{B}_1 := \check{B}_1 U$, this is illustrated in the diagram below

$$\begin{tikzcd} \mathscr{N}(\bar{B}_1)^\perp \arrow[r, "\check{B}_1"] & \mathbb{R}^x \\ \mathbb{R}^b \arrow[u, "U"] \arrow[ur, "\acute{B}_1"'] & \end{tikzcd}$$

We can obtain $\acute{B}_1$ explicitly by

$$\begin{aligned} \acute{B}_1 = \check{B}_1 U &= \bar{B}_1 |_{\mathscr{N}(\bar{B}_1)^\perp} \bar{B}_1^* T_B^* \begin{bmatrix} \Sigma_b^{-1/2} \\ 0 \end{bmatrix} \\ &= \bar{B}_1 \bar{B}_1^* T_B^* \begin{bmatrix} \Sigma_b^{-1/2} \\ 0 \end{bmatrix} = T_B^* \begin{bmatrix} \Sigma_b^{-1/2} \\ 0 \end{bmatrix}. \end{aligned}$$

Similarly for the output space, we have

$$\begin{aligned} \mathscr{R}(\bar{C}_1, \bar{D}_{12}) &= \mathscr{R}([\bar{C}_1 \ \bar{D}_{12}]) \\ &= \mathscr{R}\left([\bar{C}_1 \ \ \bar{D}_{12}] \begin{bmatrix} \bar{C}_1^* \\ \bar{D}_{12}^* \end{bmatrix} [\bar{C}_1 \ \ \bar{D}_{12}]\right) \\ &= \mathscr{R}\left([\bar{C}_1 \ \ \bar{D}_{12}] T_{CD}^* \begin{bmatrix} \Sigma_{cd}^{-1/2} \\ 0 \end{bmatrix}\right). \end{aligned}$$

So define the operator $V: \mathbb{R}^{cd} \rightarrow \mathscr{R}(\bar{C}_1, \bar{D}_{12}) \subset L^2[0, \tau]$ by

$$V := [\bar{C}_1 \ \ \bar{D}_{12}] T_{CD}^* \begin{bmatrix} \Sigma_{cd}^{-1/2} \\ 0 \end{bmatrix}.$$

As before, V is an isometry onto $\mathscr{R}(\bar{C}_1, \bar{D}_{12})$. Now $V^*|_{\mathscr{R}(\bar{C}_1, \bar{D}_{12})}$ is an isometric isomorphism from $\mathscr{R}(\bar{C}_1, \bar{D}_{12})$ to $\mathbb{R}^{cd}$, so define

$$\acute{C}_1 := V^*|_{\mathscr{R}(\bar{C}_1, \bar{D}_{12})} \check{C}_1; \quad \acute{D}_{12} := V^*|_{\mathscr{R}(\bar{C}_1, \bar{D}_{12})} \check{D}_{12}$$

which can be evaluated explicitly by

$$\begin{aligned} [\acute{C}_1 \ \ \acute{D}_{12}] &= V^*|_{\mathscr{R}(\bar{C}_1, \bar{D}_{12})} [\check{C}_1 \ \ \check{D}_{12}] \\ &= V^*|_{\mathscr{R}(\bar{C}_1, \bar{D}_{12})} \Pi_{\mathscr{R}(\bar{C}_1, \bar{D}_{12})} [\bar{C}_1 \ \ \bar{D}_{12}] \\ &= V^*[\bar{C}_1 \ \ \bar{D}_{12}] = [\Sigma_{cd}^{1/2} \ \ 0] T_{CD}. \end{aligned}$$

Thus the new system $\acute{G}$ can be written as

$$\acute{G} = \left[\begin{array}{c|cc} \bar{A} & \acute{B}_1 & \bar{B}_2 \\ \hline \acute{C}_1 & 0 & \acute{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} \bar{A} & \check{B}_1 U & \bar{B}_2 \\ \hline V^* \check{C}_1 & 0 & V^* \check{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right].$$

And when connected in feedback with any C

$$\mathscr{F}(\acute{G}, C) = V^* \mathscr{F}(\check{G}, C) U;$$

$$\mathscr{F}(\check{G}, C): l_{\mathscr{N}(\bar{B}_1)^\perp} \rightarrow l_{\mathscr{R}(\bar{C}_1, \bar{D}_{12})}, \quad \mathscr{F}(\acute{G}, C): l_{\mathbb{R}^b} \rightarrow l_{\mathbb{R}^{cd}}.$$

The fact that U and V are isometries allows us to conclude the first equality in

$$\|\mathscr{F}(\acute{G}, C)\| = \|\mathscr{F}(\check{G}, C)\| = \|\mathscr{F}(\bar{G}, C)\|$$

the second equality follows from the discussion before the theorem.

Finally, the equivalence of the internal stability of $\mathscr{F}(\acute{G}, C)$ and $\mathscr{F}(\bar{G}, C)$ is immediate since they both have the same "A" matrix (internal stability is determined by the A, B_2, C_2, D_{22} matrices of the plant and by C, all of which are the same in both $\mathscr{F}(\acute{G}, C)$ and $\mathscr{F}(\bar{G}, C)$). ■

The preceding theorem offers a solution to the $\mathscr{H}^\infty$ problem for an infinite-dimensional system of the type where the D_{11} operator is zero. Recall that the hybrid system problem is equivalent to that for $\tilde{G}$, and the D_{11} operator in $\tilde{G}$ is $\hat{D}_{11}$ which comes from the lifting of the original G_{11}. $\hat{D}_{11}$ can only be zero if G_{11} is zero, and this is rarely the case in most problems. However, $\tilde{G}$ can be reduced to a $\bar{G}$ whose D_{11} operator is zero, such that $\|\mathscr{F}(\tilde{G}, C)\| < 1$ if and only if $\|\mathscr{F}(\bar{G}, C)\| < 1$. This reduction combined with the previous theorem, will provide a complete solution to the original problem.

To accomplish this "removal" of $\hat{D}_{11}$, we use an operator-valued version of "loop-shifting" (see [22]), and for this we need Lemma 5 below, which is an operator-valued version of the Redheffer lemma [6, Lemma 15] [22, Lemma 2]. To begin with, let T be any operator on $L^2[0, \tau]$ such that $\|T\| < 1$, it follows that the operators $(I - T^*T)^{1/2}$ and $(I - TT^*)^{1/2}$ exist and are positive definite. It also follows

that the operator Θ on $L^2[0,\tau] \oplus L^2[0,\tau]$ defined by

$$\Theta := \begin{bmatrix} -T & (I - TT^*)^{1/2} \\ (I - T^*T)^{1/2} & T^* \end{bmatrix} \tag{26}$$

is unitary.

Lemma 5: Let

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}$$

be the unitary operator defined by (26). Let

$$H = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be an $l_{L^2[0,\tau]}$ system. Then the following are equivalent (see Fig. 8)

i) H is internally stable and $\|H\| < 1$;

ii) $\mathcal{F}(\Theta, H)$ is internally stable and $\|\mathcal{F}(\Theta, H)\| < 1$.

Proof: First, note that a realization for $\mathcal{F}(\Theta, H)$ is given by

$$\mathcal{F}(\Theta, H) = \left[\begin{array}{c|c} A + BR^{-1}\theta_{22}C & BR^{-1}\theta_{21} \\ \hline \theta_{12}S^{-1}C & \theta_{11} + \theta_{12}DR^{-1}\theta_{21} \end{array}\right]$$

where $R = (I - \theta_{22}D)$ and $S = (I - D\theta_{22})$.

i) ⇒ ii): For internal stability we need to show that $A_H := A + BR^{-1}\theta_{22}C$ has all its eigenvalues in the open unit disk. Recall that $\|\theta_{22}\| = \|T\| < 1$, and since $\|H\| < 1$, then $\|D\| < 1$, thus $(I - \theta_{22}D)^{-1}$ exists and $\mathcal{F}(\Theta, H)$ is well posed. It is true that A_H is the "A" matrix of the system $(I - \theta_{22}H)^{-1}$. Now since $\|H\|_{\mathcal{H}^\infty} < 1$ and $\|\theta_{22}\| \leq 1$, then $\|\theta_{22}H(z)\| < 1$ for $\{|z| \leq 1\}$ (where $H(z)$ is the z-transform of H). Thus, $(I - \theta_{22}H(z))^{-1}$ exists for $\{|z| \leq 1\}$ and therefore $(I - zA_H)^{-1}$ exists for $\{|z| \leq 1\}$ implying that all the eigenvalues of A_H are in the open unit disk.

For the norm of $\mathcal{F}(\Theta, H)$, note that Θ unitary implies (Fig. 8)

$$\|r\|^2 + \|u\|^2 = \|v\|^2 + \|y\|^2 \Rightarrow \|r\|^2 - \|v\|^2 = \|y\|^2 - \|u\|^2. \tag{27}$$

Therefore

$$\|H\| < 1 \Rightarrow \|y\|^2 - \|u\|^2 < 0 \Rightarrow \|r\|^2 - \|v\|^2 < 0 \Rightarrow \|\mathcal{F}(\Theta, H)\| < 1. \tag{28}$$

ii) ⇒ i): Define

$$\Theta' := \begin{bmatrix} T & (I - T^*T)^{1/2} \\ (I - TT^*)^{1/2} & -T^* \end{bmatrix}.$$

Θ' is unitary and $\|\theta'_{22}\| < 1$, note that these correspond to

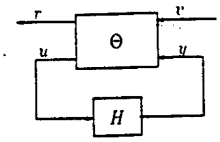

Fig. 8. $\mathcal{F}(\Theta, H)$.

the only facts about Θ used in the first part of the proof. It is straightforward to verify that $\mathcal{F}(\Theta', \mathcal{F}(\Theta, H)) = H$, therefore this direction follows from the first part of the proof with Θ replaced by Θ' and H replaced by $\mathcal{F}(\Theta, H)$. ■

We will use this lemma on $\tilde{G}$ to remove the $\tilde{D}_{11}$ operator by the proper choice of Θ. First, recall that the objective is to find C such that $\|\mathcal{F}(\tilde{G}, C)\| < 1$. Recall the realization of $\tilde{G}$

$$\tilde{G} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \tilde{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right].$$

A straightforward manipulation of this realization and that of a C shows that the "D" operator in $\mathcal{F}(\tilde{G}, C)$ is $\hat{D}_{11}$. Now by the definition of the $\mathcal{H}^\infty$ norm

$$\|\mathcal{F}(\tilde{G}, C)\|_{\mathcal{H}^\infty} = \sup_{|z|<1} \|\mathcal{F}(\tilde{G}, C)(z)\| \geq \|\mathcal{F}(\tilde{G}, C)(0)\| = \|\hat{D}_{11}\|.$$

This inequality implies that $\|\hat{D}_{11}\| < 1$ is a necessary condition for $\|\mathcal{F}(\tilde{G}, C)\| < 1$, we assume this from now on.

Given that $\|\hat{D}_{11}\| < 1$, we form the unitary operator matrix

$$\Theta = \begin{bmatrix} -\hat{D}_{11} & (I - \hat{D}_{11}\hat{D}_{11}^*)^{1/2} \\ (I - \hat{D}_{11}^*\hat{D}_{11})^{1/2} & \hat{D}_{11}^* \end{bmatrix}.$$

Putting $\tilde{G}$ and Θ in feedback as shown in Fig. 9. we obtain the new system $\bar{G}$ given by

$$\bar{G} = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & 0 & \bar{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} \hat{A} + \hat{B}_1\hat{D}_{11}^*P^{-1}\hat{C}_1 & \hat{B}_1(I - \hat{D}_{11}^*\hat{D}_{11})^{-1/2} & \hat{B}_1\hat{D}_{11}^*P^{-1}\tilde{D}_{12} + \tilde{B}_2 \\ \hline (I - \hat{D}_{11}\hat{D}_{11}^*)^{-1/2}\hat{C}_1 & 0 & (I - \hat{D}_{11}\hat{D}_{11}^*)^{-1/2}\tilde{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right]$$

where $P = (I - \hat{D}_{11}\hat{D}_{11}^*)$.

Lemma 5 states that $\|\mathcal{F}(\tilde{G}, C)\| < 1$ if and only if $\|\mathcal{F}(\Theta, \mathcal{F}(\tilde{G}, C))\| = \|\mathcal{F}(\bar{G}, C)\| < 1$. But now $\bar{G}$ is in the form needed by Theorem 4 to reduce the problem to a finite-dimensional one. We summarize this in the next theorem.

Theorem 6: Given the infinite-dimensional system $\tilde{G}$ defined by

$$\tilde{G} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \tilde{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right]$$

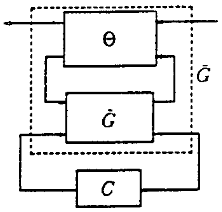

Fig. 9. $\mathcal{F}(\Theta, \mathcal{F}(\tilde{G}, C)) = \mathcal{F}(\bar{G}, C)$.

with $\|\hat{D}_{11}\| < 1$. Form the matrices

$$\hat{B}_1\left(I - \hat{D}_{11}^*\hat{D}_{11}\right)^{-1}\hat{B}_1^* = T_B^*\begin{bmatrix} \Sigma_b & 0 \\ 0 & 0 \end{bmatrix} T_B;$$

$$\begin{bmatrix} \hat{C}_1^* \\ \tilde{D}_{12}^* \end{bmatrix}\left(I - \hat{D}_{11}\hat{D}_{11}^*\right)^{-1}\left[\hat{C}_1 \tilde{D}_{12}\right] = T_{CD}^*\begin{bmatrix} \Sigma_{cd} & 0 \\ 0 & 0 \end{bmatrix} T_{CD}$$

where Σ_b, Σ_{cd} are diagonal and nonsingular, and T_B, T_{CD} are finite matrices. Define the finite-dimensional system

$$\acute{G} = \left[\begin{array}{c|cc} \acute{A} & \acute{B}_1 & \acute{B}_2 \\ \hline \acute{C}_1 & 0 & \acute{D}_{12} \\ \acute{C}_2 & 0 & 0 \end{array}\right] : \begin{array}{c} l_{\mathbb{R}^b} \\ \oplus \\ l_{\mathbb{R}^u} \end{array} \to \begin{array}{c} l_{\mathbb{R}^{cd}} \\ \oplus \\ l_{\mathbb{R}^y} \end{array}$$

where

$$\acute{B}_1 := T_B^*\begin{bmatrix} \Sigma_b^{1/2} \\ 0 \end{bmatrix}, \quad \left[\acute{C}_1 \quad \acute{D}_{12}\right] := \left[\Sigma_{cd}^{1/2} \quad 0\right] T_{CD}$$

$$\acute{A} := \hat{A} + \hat{B}_1\hat{D}_{11}^*\left(I - \hat{D}_{11}\hat{D}_{11}^*\right)^{-1}\hat{C}_1;$$

$$\acute{B}_2 := \hat{B}_1\hat{D}_{11}^*\left(I - \hat{D}_{11}\hat{D}_{11}^*\right)^{-1}\tilde{D}_{12} + \tilde{B}_2; \quad \acute{C}_2 := \tilde{C}_2.$$

Then the following are equivalent:

i) $\mathcal{F}(\tilde{G}, C)$ is internally stable and $\|\mathcal{F}(\tilde{G}, C)\| < 1$.

ii) $\mathcal{F}(\acute{G}, C)$ is internally stable and $\|\mathcal{F}(\acute{G}, C)\| 1$.

Remarks:

i) Note that even though quantities like $(I - \hat{D}_{11}\hat{D}_{11}^*)^{1/2}$ appear in Θ and $\bar{G}$, this operator square root does not need to be evaluated since in the final equivalence to $\acute{G}$ it does not appear.

ii) To apply Theorem 6, one needs to compute the following operator compositions (which are finite matrices)

$$\hat{B}_1\left(I - \hat{D}_{11}^*\hat{D}_{11}\right)^{-1}\hat{B}_1^*;$$

$$\begin{bmatrix} \hat{C}_1^* \\ \tilde{D}_{12}^* \end{bmatrix}\left(I - \hat{D}_{11}\hat{D}_{1a}^*\right)^{-1}\left[\hat{C}_1 \tilde{D}_{12}\right] = T_{CD}^*$$

$$\hat{A} + \hat{B}_1\hat{D}_{11}^*\left(I - \hat{D}_{11}\hat{D}_{11}^*\right)^{-1}\hat{C}_1;$$

$$\hat{B}_1\hat{D}_{11}^*\left(I - \hat{D}_{11}\hat{D}_{11}^*\right)^{-1}\tilde{D}_{12} + \tilde{B}_2.$$

In the next section, the required operator compositions are done analytically, and formulas for the resulting matrices are given in terms of the matrices of the realization of the original continuous-time system G. The formulas involve only matrix operations such as exponentiation and inversion.

iii) The factors $\Sigma_b, T_B, \Sigma_{cd}, T_{CD}$ can be obtained by diagonalizing the two symmetric matrices using well known algorithms such as in [13].

iv) In the next section, we will also give methods for checking the condition $\|\hat{D}_{11}\| < 1$. This condition has an interesting connection with an $\mathcal{H}^\infty$ problem for delay systems.

V. The Operator $\hat{D}_{11}$ and Other Explicit Formulas

Theorem 6 yields the equivalent finite-dimensional problem given the original infinite-dimensional system $\tilde{G}$. $\tilde{G}$ is given explicitly below (see also Fig. 5 and (18)) from the matrices of the realization of the original continuous time generalized plant G

$$\tilde{G} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \tilde{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} e^{A\tau} & e^{A(\tau-s)}B_1 & \Psi(\tau)B_2 \\ \hline C_1e^{At} & C_1e^{A(t-s)}\mathbf{1}_{(t-s)}B_1 & C_1\Psi(t)B_2 \\ C_2 & 0 & 0 \end{array}\right]. \tag{29}$$

Where we have assumed, for simplicity, that the matrices $D_{11} = D_{12} = 0$. For the remainder of this section we will also assume that (A, B_1) is controllable and (C_1, A) is observable.

To carry out the explicit computations called for by Theorem 6 we need to examine carefully the operators $(I - \hat{D}_{11}^*\hat{D}_{11})^{-1}$ and $(I - \hat{D}_{11}\hat{D}_{11}^*)^{-1}$. Recall that $\hat{D}_{11}$ is the "truncation" of G_{11}, that is $\hat{D}_{11} = \Pi_{L^2[0,\tau]}G_{11}|_{L^2[0,\tau]}$. The easiest way to deal with this operator is to consider the associated system of differential equations over the finite-time horizon $[0, \tau]$. Note that regardless of whether G_{11} is stable or not, $\hat{D}_{11}$ is an $L^2[0,\tau]$ stable operator. The relation $f = \hat{D}_{11}u$ is equivalent to the following system of differential equations:

$$\begin{aligned} \dot{x}_1(t) &= Ax_1(t) + B_1u(t) \\ f(t) &= C_1x_1(t); \quad x_1(0) = 0, \qquad 0 \le t \le \tau. \end{aligned} \tag{30}$$

It is easy to verify that the adjoint operator is given by the adjoint differential equation, that is, $y = \hat{D}_{11}^*f$ means

$$\begin{aligned} \dot{x}_2(t) &= -A'x_2(t) - C_1'f(t) \\ y(t) &= B_1'x_2(t); \quad x_2(\tau) = 0, \qquad 0 \le t \le \tau. \end{aligned} \tag{31}$$

Combining (30) and (31), it follows that the operator composition $y = (I - \hat{D}_{11}^*\hat{D}_{11})u$ is given by

$$\begin{bmatrix} \dot{x}_2(t) \\ \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} -A' & -C_1'C_1 \\ 0 & A \end{bmatrix}\begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -B_1 \end{bmatrix}u(t) \tag{32}$$

$$y(t) = \left[B_1' \quad 0\right]\begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} + u(t);$$

$$\begin{bmatrix} x_2(\tau) \\ x_1(0) \end{bmatrix} = 0; \qquad 0 \le t \le \tau.$$

Note the two-point boundary condition on the states. The inverse (if it exists), can be found by rewriting the equations for u in terms of y, yielding

$$\begin{bmatrix} \dot{x}_2(t) \\ \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} -A' & -C_1'C_1 \\ B_1B_1' & A \end{bmatrix} \begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} y(t) \quad (33)$$

$$u(t) = [B_1' \quad 0] \begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} + y(t);$$

$$\begin{bmatrix} x_2(\tau) \\ x_1(0) \end{bmatrix} = 0; \qquad 0 \le t \le \tau.$$

In contrast to (32), this system of equations may not have a solution. It has a solution when the two-point boundary values are well posed, and this happens exactly when the operator $(I - \hat{D}_{11}^* \hat{D}_{11})$ is invertible [12]. The necessary and sufficient condition for this is as follows; form the matrix

$$\Gamma(t) = \begin{bmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ \Gamma_{21}(t) & \Gamma_{22}(t) \end{bmatrix} = \exp\left\{ \begin{bmatrix} -A' & -C_1'C_1 \\ B_1B_1' & A \end{bmatrix} t \right\} \quad (34)$$

then (33) has a solution, or equivalently, $(I - \hat{D}_{11}^* \hat{D}_{11})$ is invertible if and only if $\Gamma_{11}(\tau)$ (or equivalently, $\Gamma_{22}(\tau)$) is invertible ([27], [12]). We remark here that a similar argument is used in [27] to obtain a condition for a given σ to be a singular value of an operator like $\hat{D}_{11}$, there the invertibility of $(\sigma^2 I - \hat{D}_{11}^* \hat{D}_{11})$ is in question, and a condition similar to the one above is given.

The standing assumption here is that $\| \hat{D}_{11} \| < 1$, since, as remarked in the previous section, this is a necessary condition for the existence of a C such that $\| \mathscr{F}(\tilde{G}, C) \| < 1$. This assumption guarantees that the operator $(I - \hat{D}_{11}^* \hat{D}_{11})$ is invertible, implying that $\Gamma_{11}(\tau)$ and $\Gamma_{22}(\tau)$ are also invertible.

To find the kernel representation of the operator $(I - \hat{D}_{11}^* \hat{D}_{11})^{-1}$, we find the solution of the differential equation (33) as a function of the input. Let $\Gamma(t)$ be as in (34), it is the state transition matrix for the system (33). From the variations of constants formula, given any input $y(t)$, the states are given by

$$\begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} = \Gamma(t - t_o) \begin{bmatrix} x_2(t_o) \\ x_2(t_o) \end{bmatrix} + \int_{t_o}^{t} \Gamma(t - s) \begin{bmatrix} 0 \\ B_1 \end{bmatrix} y(s)\, ds. \quad (35)$$

Using the two boundary conditions $x_2(\tau) = 0$ and $x_1(0) = 0$ twice in (35) and subtracting the resulting equations we get

$$\begin{bmatrix} x_2(0) \\ x_1(\tau) \end{bmatrix} = \begin{bmatrix} -\Gamma_{11}^{-1}(\tau) & 0 \\ \Gamma_{21}(\tau)\Gamma_{11}^{-1}(\tau) & I \end{bmatrix} \cdot \int_0^{T} \Gamma(\tau - s) \begin{bmatrix} 0 \\ B_1 \end{bmatrix} y(s)\, ds. \quad (36)$$

Using the expression for $x_2(0)$ and (35) (with $t_o = 0$), we obtain the output $u(t)$ as

$$u(t) = [B_1' \quad 0] \left(\Gamma(t) \begin{bmatrix} -\Gamma_{11}^{-1}(\tau) & 0 \\ 0 & 0 \end{bmatrix} \cdot \int_0^{\tau} \Gamma(\tau - s) \begin{bmatrix} 0 \\ B_1 \end{bmatrix} y(s)\, ds \cdot + \int_0^{t} \Gamma(t - s) \begin{bmatrix} 0 \\ B_1 \end{bmatrix} y(s)\, ds \right) + y(t).$$

Thus the operator $(I - \hat{D}_{11}^* \hat{D}_{11})^{-1}$ is given by the kernel

$$(I - \hat{D}_{11}^* \hat{D}_{11})^{-1}(t, s) = [0 \quad C_1] \left\{ \Gamma(t) \begin{bmatrix} \Gamma_{11}^{-1}(\tau) & 0 \\ 0 & 0 \end{bmatrix} \Gamma(\tau - s) \begin{bmatrix} C_1' \\ 0 \end{bmatrix} + \mathbf{1}_{(t-s)} \Gamma(t - s) \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \right\} + I\delta(t - s). \quad (37)$$

As for the operator $(I - \hat{D}_{11} \hat{D}_{11}^*)^{-1}$, a similar manipulation of the differential equation and its adjoint as above yields the following kernel

$$(I - \hat{D}_{11} \hat{D}_{11}^*)^{-1}(t, s) = [0 \quad C_1] \left\{ \Gamma(t) \begin{bmatrix} \Gamma_{11}^{-1}(\tau) & 0 \\ 0 & 0 \end{bmatrix} \Gamma(\tau - s) \begin{bmatrix} C_1' \\ 0 \end{bmatrix} - \mathbf{1}_{(t-s)} \begin{bmatrix} C_1' \\ 0 \end{bmatrix} \right\} + I\delta(t - s).$$

These kernel representations can be used to compute the operator compositions required for the application of Theorem 6. The computations are rather lengthy, here are the final formulas (see the Appendix for the details):

$$\hat{B}_1 (I - \hat{D}_{11}^* \hat{D}_{11})^{-1} \hat{B}_1^* = \Gamma_{21}(\tau)\Gamma_{11}^{-1}(\tau) \quad (38)$$

$$\hat{C}_1^* (I - \hat{D}_{11} \hat{D}_{11}^*)^{-1} \hat{C}_1 = -\Gamma_{11}^{-1}(\tau)\Gamma_{12}(\tau) \quad (39)$$

$$\hat{A} + \hat{B}_1 \hat{D}_{11}^* (I - \hat{D}_{11} \hat{D}_{11}^*)^{-1} \hat{C}_1 = \Gamma_{22}(\tau) - \Gamma_{21}(\tau) \cdot \Gamma_{11}^{-1}(\tau)\Gamma_{12}(\tau) \quad (40)$$

$$\hat{B}_1 \hat{D}_{11}^* (I - \hat{D}_{11} \hat{D}_{11}^*)^{-1} \tilde{D}_{12} = [\Phi_{22}(\tau) - \Psi(\tau) - \Gamma_{21}(\tau)\Gamma_{11}^{-1}(\tau)\Phi_{12}(\tau)] B_2 \quad (41)$$

$$\hat{C}_1^* (I - \hat{D}_{11} \hat{D}_{11}^*)^{-1} \tilde{D}_{12} = -\Gamma_{11}^{-1}(\tau)\Phi_{12}(\tau) B_2 \quad (42)$$

$$\tilde{D}_{12}^* (I - \hat{D}_{11} \hat{D}_{11}^*)^{-1} \tilde{D}_{12} = B_2' [\Omega_{12}(\tau) - \Phi_{11}(\tau) \cdot \Gamma_{11}^{-1}(\tau)\Phi_{12}(\tau)] B_2 \quad (43)$$

where $\Psi(t) := \int_0^t e^{As}\, ds$, and the matrices $\Phi(\tau), \Omega(\tau)$ are defined by

$$\Phi(t) := \int_0^t \Gamma(s)\, ds$$

$$\Omega(t) := \int_0^t \left(\int_0^s \Gamma(r)\, dr \right) ds \quad (44)$$

and are partitioned conformably with $\Gamma(t)$. We note that the integrations required to form Ψ, Φ, Ω can be done using the formula

$$\int_0^t e^{As}\, ds = [I \quad 0] e^{\begin{bmatrix} A & I \\ 0 & 0 \end{bmatrix} t} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{45}$$

which is true for any matrix A. Thus $\Psi(\tau)$, $\Phi(\tau)$ can be computed each by performing matrix exponentiations, and using (45) twice in (44), $\Omega(\tau)$ is computed by performing one matrix exponentiation.

With these formulas and Theorem 6, the equivalent finite-dimensional problem can be obtained from the realization of the original plant in the hybrid system.

Finally, note that from Theorem 6, a necessary condition for the existence of a controller that constrains the closed-loop norm to be less than 1 is that $\|\hat{D}_{11}\| < 1$, (this condition also guarantees the invertibility of $\Gamma_{11}(\tau)$), which leads us to the question of how this condition could be checked. There is a connection between $\|\hat{D}_{11}\|$ and the value of a certain $\mathscr{H}^\infty$ problem for delay systems. Recall that $\hat{D}_{11} = \Pi_{L^2[0,\tau]} G_{11} \mid_{L^2[0,\tau]}$. If G_{11} is stable, it is an application of Sarason's result [23] (see also [7]) to show that

$$\|\hat{D}_{11}\| = \|\Pi_{L^2[0,\tau]} G_{11} \mid_{L^2[0,\tau]}\| = \inf_{Q \in \mathscr{H}^\infty} \|G_{11} - e^{-s\tau} Q\|. \tag{46}$$

Thus the norm of $\hat{D}_{11}$ is the value of a certain sensitivity minimization problem for a plant with pure delay. Using so-called "skew Toeplitz theory" the computation of this norm can be reduced to finding the singular values of a certain finite matrix. See [8] for all the details. (Software for this purpose already exists at the University of Minnesota and Honeywell, SRC.) Consequently, one can explicitly compute the norm $\|\hat{D}_{11}\|$.

For the case when G_{11} is not necessarily stable a different method can be used to compute $\|\hat{D}_{11}\|$. Let $\mathscr{H}_n$ and $\mathscr{S}_n$ be the following operators defined between $L^2[0,\tau]$ and $\mathbb{R}^n$ ($\mathbb{R}^n$ with the euclidean norm):

$$\mathscr{S}_n : L^2[0,\tau] \to \mathbb{R}^n \quad (\mathscr{S}_n u)(i) = u\left(\frac{\tau}{n} i\right); \qquad u \in L^2[0,\tau]$$

$$\mathscr{H}_n : \mathbb{R} \to L^2[0,\tau] \quad (\mathscr{H}_n u)(t) = u\left(\left\lfloor \frac{tn}{\tau} \right\rfloor\right); \quad \{u(i)\} \in \mathbb{R}^n$$

(strictly speaking, $\mathscr{S}_n$ is not an operator on $L^2[0,\tau]$ but on the subspace of left and right continuous functions, this distinction is irrelevant here since in our use of it below, $\mathscr{S}_n$ operates only on continuous signals), the above operators can be thought of as "fast" sample and hold operators.

We now form the matrix $\mathscr{S}_n \hat{D}_{11} \mathscr{H}_n : \mathbb{R}^{n \times w} \to \mathbb{R}^{n \times z}$ (recall that w, z are the dimensions of the signals w and z, respectively). The matrix can be explicitly computed as follows, denote by $(\mathscr{S}_n \hat{D}_{11} \mathscr{H}_n)_{i,j}$, the i,jth block of size $z \times w$, then

$$(\mathscr{S}_n \hat{D}_{11} \mathscr{H}_n)_{i,j} = \begin{cases} C_1 e^{A(\tau/n)(i-j)} \Psi(\tau/n) B_1 & \text{for } i - j > 1 \\ 0 & \text{for } i - j \le 0 \end{cases}.$$

Now, it can be shown that $\|\mathscr{S}_n \hat{D}_{11} \mathscr{H}_n\| \xrightarrow{n \to \infty} \|\hat{D}_{11}\|$, where $\|\mathscr{S}_n \hat{D}_{11} \mathscr{H}_n\|$ is the induced norm over Euclidean space (i.e., the maximum singular value). Thus we can compute $\|\hat{D}_{11}\|$ by taking n large and computing the maximum singular value of the matrix $\mathscr{S}_n \hat{D}_{11} \mathscr{H}_n$.

Appendix
Outline of the Derivation of the Formulas (38)–(43)

The formulas for the matrices (38)–(43) involve the compositions of the appropriate operators. The compositions are performed by integrating the kernel representations of the operators against each other, and the given formulas are the results of the explicit evaluation of these integrations.

(38)–(40):

We give an outline of the deviation of (38), the derivations of (39)–(40) are entirely similar and are therefore omitted.

We first determine the operator $(I - \hat{D}_{11}^* \hat{D}_{11})^{-1} \hat{B}_1^*$, since it is an operator from $\mathbb{R}^n$ to $L^2[0,\tau]$, it is given by a kernel which is function of one variable, specifically

$$\left((I - \hat{D}_{11}^* \hat{D}_{11})^{-1} \hat{B}_1^*\right)(t) = \int_0^\tau (I - \hat{D}_{11}^* \hat{D}_{11})^{-1}(t,s) \hat{B}_1^*(s)\, ds.$$

From the formulas for the kernels of $(I - \hat{D}_{11}^* \hat{D}_{11})^{-1}$ (37) and $\hat{B}_1^*$ (29), we compute

$$\begin{aligned} \left((I - \hat{D}_{11}^* \hat{D}_{11})^{-1} \hat{B}_{11}^*\right)(t) &= [B_1' \quad 0] \Bigg\{ \Gamma(t) \begin{bmatrix} -\Gamma_{11}^{-1}(\tau) & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad \cdot \int_0^\tau e^{\begin{bmatrix} -A' & -C_1'C_1 \\ B_1B_1' & A \end{bmatrix}(\tau - s)} \begin{bmatrix} 0 \\ B_1 B_1' \end{bmatrix} e^{A'(\tau - s)}\, ds \\ &\quad + \int_0^t e^{\begin{bmatrix} -A' & -C_1'C_1 \\ B_1B_1' & A \end{bmatrix}(t - s)} \begin{bmatrix} 0 \\ B_1 B_1' \end{bmatrix} e^{A'(\tau - s)}\, ds \Bigg\} \\ &\quad + B_1' e^{A'(\tau - t)}. \end{aligned}$$

The integrals in the equation above can be explicitly evaluated by noting that

$$\frac{d}{ds}\left\{ e^{-\begin{bmatrix} -A' & -C_1'C_1 \\ B_1B_1' & A \end{bmatrix} s} \begin{bmatrix} I \\ 0 \end{bmatrix} e^{-A's} \right\} = e^{-\begin{bmatrix} -A' & -C_1'C_1 \\ B_1B_1' & A \end{bmatrix} s} \begin{bmatrix} 0 \\ -B_1 B_1' \end{bmatrix} e^{-A's}. \tag{47}$$

After evaluation of the integrals, some terms cancel, and we obtain

$$\left((I - \hat{D}_{11}^* \hat{D}_{11})^{-1} \hat{B}_1^*\right)(t) = [B_1' \quad 0] e^{\begin{bmatrix} -A' & -C_1'C_1 \\ B_1B_1' & A \end{bmatrix} t} \begin{bmatrix} \Gamma_{11}^{-1}(\tau) \\ 0 \end{bmatrix}.$$

To evaluate the matrix $\hat{B}_1(I - \hat{D}_{11}^* \hat{D}_{11})^{-1} \hat{B}_1^*$, we inte-

grate the kernels of the operators $\hat{B}_1$ and $(I - \hat{D}_{11}^* \hat{D}_{11})^{-1}\hat{B}_1^*$

$$\begin{aligned}\hat{B}_1(I-\hat{D}_{11}^*\hat{D}_{11})^{-1}\hat{B}_1^* &= \int_0^T \hat{B}_1(t)\left((I-\hat{D}_{11}^*\hat{D}_{11})^{-1}\hat{B}_1^*\right)(t)\,dt\\ &= \int_0^T e^{A(\tau-t)}[B_1B_1' \quad 0]\, e^{\begin{bmatrix}-A' & -C_1'C_1\\ B_1B_1' & A\end{bmatrix}t} dt \begin{bmatrix}\Gamma_{11}^{-1}(\tau)\\ 0\end{bmatrix}.\end{aligned}$$

This integral can also be evaluated explicitly using an identity similar to (47), and we obtain

$$\begin{aligned}\hat{B}_1(I-\hat{D}_{11}^*\hat{D}_{11})^{-1}\hat{B}_1^* &= [0 \quad I]\, e^{\begin{bmatrix}-A' & -C_1'C_1\\ B_1B_1' & A\end{bmatrix}\tau}\begin{bmatrix}I\\ 0\end{bmatrix}\Gamma_{11}^{-1}(\tau)\\ &= \Gamma_{21}(\tau)\Gamma_{11}^{-1}(\tau).\end{aligned}$$

The derivations of (39) and (40) are very similar to the above, the only nonroutine steps being several uses of the identity (47).

(41)-(43):

As before we only outline the derivation of (42), the other two being very similar.

The derivation of (42) is slightly more complicated than what we have already seen because of the operator $\tilde{D}_{12}$. First, recall that the kernel of $\tilde{D}_{12}$ is given by

$$\tilde{D}_{12}(t) = C_1\left(\int_0^t e^{Ar}\,dr\right)B_2.$$

A simple change of variables shows that

$$\tilde{D}_{12}(t) = C_1\left(\int_0^t e^{Ar}dr\right)B_2 = C_1\left(\int_0^t e^{A(t-\hat{r})}\,d\hat{r}\right)B_2.$$

Now we compute

$$\begin{aligned}&\left((I-\hat{D}_{11}\hat{D}_{11}^*)^{-1}\tilde{D}_{12}\right)(t)\\ &= [0 \quad C_1']\left\{\Gamma(t)\begin{bmatrix}\Gamma_{11}^{-1}(\tau) & 0\\ 0 & 0\end{bmatrix}\right.\\ &\quad\cdot\int_0^T e^{\begin{bmatrix}-A' & -C_1'C_1\\ B_1B_1' & A\end{bmatrix}(\tau-s)}\begin{bmatrix}C_1'C_1\\ 0\end{bmatrix}\int_0^s e^{A(s-r)}\,dr\,B_2\,ds\\ &\quad\left.-\int_0^t e^{\begin{bmatrix}-A' & -C_1'C_1\\ B_1B_1' & A\end{bmatrix}(t-s)}\begin{bmatrix}C_1'C_1\\ 0\end{bmatrix}\int_0^s e^{A(s-r)}\,dr\,B_2\,ds\right\}\\ &\quad + C_1\int_0^t e^{A(t-r)}\,dr\,B_2\end{aligned}$$

integrating with respect to the variable s first

$$\begin{aligned}&= [0 \quad C_1']\left\{\Gamma(t)\begin{bmatrix}\Gamma_{11}^{-1}(\tau) & 0\\ 0 & 0\end{bmatrix}\right.\\ &\quad\cdot\int_0^T\int_r^T e^{\begin{bmatrix}-A' & -C_1'C_1\\ B_1B_1' & A\end{bmatrix}(\tau-s)}\begin{bmatrix}C_1'C_1\\ 0\end{bmatrix}e^{A(s-r)}\,ds\,B_2\,dr\\ &\quad\left.-\int_0^t\int_r^t e^{\begin{bmatrix}-A' & -C_1'C_1\\ B_1B_1' & A\end{bmatrix}(t-s)}\begin{bmatrix}C_1'C_1\\ 0\end{bmatrix}e^{A(s-r)}\,ds\,B_2\,dr\right\}\\ &\quad + C_1\int_0^t e^{A(t-r)}\,dr\,B_2.\end{aligned}$$

The two integrals (in s) can be evaluated explicitly using

$$\begin{aligned}\frac{d}{ds}&\left\{e^{-\begin{bmatrix}-A' & -C_1'C_1\\ B_1B_1' & A\end{bmatrix}s}\begin{bmatrix}0\\ I\end{bmatrix}e^{As}\right\}\\ &= e^{-\begin{bmatrix}-A' & -C_1'C_1\\ B_1B_1' & A\end{bmatrix}s}\begin{bmatrix}C_1'C_1\\ 0\end{bmatrix}e^{As}. \qquad (48)\end{aligned}$$

This yields (after cancellations)

$$\begin{aligned}&\left((I-\hat{D}_{11}\hat{D}_{11}^*)^{-1}\tilde{D}_{12}\right)(t)\\ &= [0 \quad C_1']\left\{-\Gamma(t)\begin{bmatrix}\Gamma_{11}^{-1}(\tau) & 0\\ 0 & 0\end{bmatrix}\right.\\ &\quad\left.\cdot\int_0^T\Gamma(\tau-r)\,dr + \int_0^t\Gamma(t-r)\,dr\right\}\begin{bmatrix}0\\ I\end{bmatrix}B_2.\end{aligned}$$

The matrix $\hat{C}_1^*(I-\hat{D}_{11}\hat{D}_{11}^*)^{-1}\tilde{D}_{12}$ is evaluated by integrating the kernel of $\hat{C}_1^*$ (that is $\{e^{A't}C_1'\}$) against the kernel above, this involves the use of an identity like (48) and switching of integrals and yields, after cancellations

$$\begin{aligned}\hat{C}_1^*(I-\hat{D}_{11}\hat{D}_{11}^*)^{-1}\tilde{D}_{12} &= -\Gamma_{11}^{-1}(\tau)[I \quad 0]\\ &\quad\cdot\int_0^T\Gamma(\tau-r)\,dr\begin{bmatrix}0\\ I\end{bmatrix}B_2\\ &= -\Gamma_{11}^{-1}(\tau)\Phi_{12}(\tau)B_2.\end{aligned}$$

The derivations of (41) and (43) are very similar to the derivation above, the only nonroutine steps being the use of identities like (48) and the switching of the order of integration in a manner very similar to that shown above.

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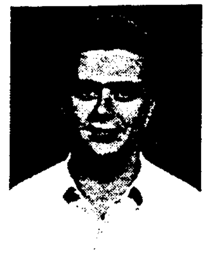

Bassam A. Bamieh (S'87–M'91) was born in Jordan in 1963. He received the B.S. degree in electrical engineering from Valparaiso University, Valparaiso, IN, in 1983, and the M.Sc. and Ph.D degrees from Rice University, Houston, TX, in 1986 and 1992, respectively.

He is currently an Assistant Professor in the Department of Electrical and Computer Engineering and in the Coordinated Science Laboratory at the University of Illinois, Urbana-Champaign. His general area of interest is control and system theory; his current areas of interest include robust control, infinite-dimensional systems, and digital control and sampled-data systems.

J. Boyd Pearson, Jr., (S'57–M'58–M'65–SM'77–F'79) received the Ph.D. degree in 1962 from Purdue University, Lafayette, IN.

He served as Assistant Professor of Electrical Engineering at Purdue from 1962 to 1965, when he joined Rice University, Houston, TX, where he is now a J.S. Abercrombie Professor.

Dr. Pearson served as President of the IEEE Control Systems Society in 1984 and received outstanding paper awards in 1985 with Bor-Chin Chang and in 1989 with Munther Dahleh.

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Minimization of the L^∞-Induced Norm for Sampled-Data Systems

Bassam Bamieh
Munther A. Dahleh
J. Boyd Pearson

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Minimization of the L^∞-Induced Norm for Sampled-Data Systems

Bassam Bamieh, *Member, IEEE*, Munther A. Dahleh, *Member, IEEE*, and J. Boyd Pearson, *Fellow, IEEE*

Abstract—The following problem is addressed: Given a continuous-time plant, with continuous-time performance objectives, expressed in terms of the L^∞-induced norm, design a digital controller that delivers or optimizes this performance. This problem differs from the standard discrete-time methods in that it takes into consideration the inter-sample behavior of the closed-loop system. The resulting closed-loop system dynamics consist of both continuous-time and discrete-time dynamics and thus such systems are known as *hybrid* systems. It is shown that given any degree of accuracy, there exists a standard discrete-time l^1 problem, which can be determined *a priori*, whose solution yields a controller that is almost optimal in terms of the hybrid L^∞-induced norm. This is accomplished by first converting the hybrid system into an *equivalent* infinite-dimensional discrete-time system using the lifting technique in continuous time, then the infinite-dimensional parts of the system which model the inter-sample dynamics are approximated. We present a thorough analysis of the approximation procedure, and show that it is convergent at the rate of $(1/n)$. Explicit bounds that are independent of the controller are obtained to characterize the approximation. Finally, it is shown that the geometry of the induced norm for the sampled-data problem is different than that of the standard l^1 norm, and hence there might not exist a linear isometry that maps the sampled-data problem exactly to a standard discrete-time problem.

I. Introduction

THIS paper is concerned with designing digital controllers for continuous-time systems to optimally achieve certain performance specifications in the presence of uncertainty. Contrary to discrete-time designs, such controllers are designed taking into consideration the inter-sample behavior of the system. Such hybrid systems are generally known as sampled-data systems, and have recently received renewed interest by the control community.

The difficulty in considering the continuous-time behavior of sampled-data systems, is that it is time varying, even when the plant and the controller are both continuous-time and discrete-time time invariant, respectively. In this paper, we consider the *standard problem with sampled-data controllers* (or the sampled-data problem, for short) shown in Fig. 1. The continuous-time controller is constrained to be sampled-data controller, that is, it is of the form $\mathscr{H}_\tau\mathscr{C}\mathscr{S}_\tau$. The generalized plant is continuous-time time invariant and C is discrete-time time invariant, $\mathscr{H}_\tau$ is a zero order hold (with period τ), and $\mathscr{S}_\tau$ is an ideal sampler (with period τ). $\mathscr{H}_\tau$ and $\mathscr{S}_\tau$ are assumed synchronized. Let $\mathscr{F}(G,\mathscr{H}_\tau C\mathscr{S}_\tau)$ denote the mapping between the exogenous input and the regulated output (i.e., w and z). $\mathscr{F}(G,\mathscr{H}_\tau C\mathscr{S}_\tau)$ is in general time varying, in fact it is τ-periodic where τ is the period of the sample and hold devices.

Sampled-data systems have been studied by many researchers in the past in the context of LQG controllers (e.g., [21]). Recently, Chen and Francis [4] studied this problem in the context of $\mathscr{H}^\infty$ control, and were able to provide a solution in the case where the regulated output is in discrete time and the exogenous input is in continuous time. The exact problem was solved in [1], [2], and independently in [13] and [22]. The L^∞-induced norm problem (the one we are concerned with in this paper) was considered in [10].

In this paper, we will use the framework developed in [1], [2], to study the l^1 sampled-data problem. Precisely, the controller is designed to minimize the induced norm of the periodic system over the space of bounded inputs (i.e., L^∞). This minimization results from posing time domain specifications and design constraints, which is quite natural for control system design. To emphasize the point made earlier, the inputs are continuous-time inputs, the errors are continuous-time errors (see Fig. 1), however the system is a hybrid system with a continuous-time plant and a discrete-time controller. The discrete-time method for l^1 designs (e.g., [5], [17], [9]), cannot handle this problem directly, and is only concerned with the performance at the sampling instants.

The solution provided in this paper is to solve the sampled-data problem by solving an (almost) equivalent discrete-time l^1 problem. While this was the approach followed in [10], the main contribution of this paper is using the lifting framework of [1], [2] to provide a thorough and more transparent analysis of the approximations involved in forming the almost equivalent problem. Furthermore, our analysis shows explicitly how the approximation procedure amounts to approximating the

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B. Bamieh is with the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801.

M. A. Dahleh is with the Laboratory of Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA.

J. B. Pearson is with the Department of Electrical and Computer Engineering, Rice University, Houston, TX 77251.

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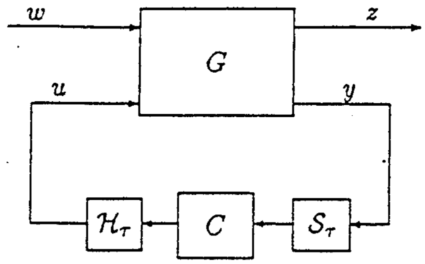

Fig. 1. Hybrid discrete/continuous-time system.

inter-sample dynamics of the hybrid system, and that the inter-sample dynamics are governed only by the plant and not the controller dynamics. We use the latter fact to derive explicit bounds on the approximation [main inequality (5)] which can be computed *a priori* and depend only on the plant. We also show that the rate of convergence of the approximation is $(1/n)$.

As already mentioned, sampled-data systems are periodic, the main theoretical tool we use for dealing with periodic systems is a *lifting* technique for continuous-time systems developed in [1], [2].[1] The technique establishes a strong correspondence between periodic systems and time invariant infinite-dimensional systems. In the next section, we briefly describe the lifting and it's application to the sampled-data problem. We then set up an equivalent infinite-dimensional problem whose solution is obtained using an approximation procedure. Formulas for the (almost) equivalent discrete-time problem are given in Section III. In the later sections, the issue of the convergence of the approximation procedure is investigated, this is done by decomposing the equivalent infinite-dimensional problem and analyzing the decomposition. In the last section, a geometric interpretation is given for the reduction of the infinite-dimensional problem, and it is compared with the $\mathscr{H}^\infty$ sampled-data problem from [1]. We also discuss possible reasons behind the fact that in the l^1 sampled-data problem (in contrast to the $\mathscr{H}^\infty$ sampled-data problem), the solutions are given by approximation, rather than exact procedures.

Finally, we note that although the closed loop, sampled-data system is periodically time varying, and thus one cannot refer to the l^1 norm of its impulse response, it is shown in [3] that the L^∞-induced norm of a periodic system can be interpreted as a type of an l^1 norm of the operator-valued "impulse response" of the lifted system. This justifies calling this problem the l^1 sampled-data problem.

II. The Lifting Technique in Sampled-Data Systems

In this section, we briefly summarize the lifting technique for continuous-time periodic systems developed in [1], [2], and apply it to the sampled-data problem. The idea of the lifting technique is to put a periodic continuous-time system in a strong correspondence with a shift invariant (i.e., discrete-time time-invariant) system, which amounts to rearranging the original system so that its periodicity can be viewed as shift invariance. To accomplish this, we first define the lifting for signals, for which the appropriate signal spaces need to be established.

For continuous-time signals, we consider the usual $L^\infty[0,\infty)$ space of essentially bounded functions [8], and it's extended version $L_e^\infty[0,\infty)$. We will also need to consider discrete-time signals that take values in a function space, for this, we define l_X to be the space of all X-valued sequences, where X is some Banach space. We define l_X^∞ as the subspace of l_X with bounded norm sequences, i.e., where for $\{f_i\} \in l_X$, the norm $\|\{f_i\}\|_{l_X^\infty} := \sup_i \|f_i\|_X < \infty$. Given any $f \in L_e^\infty[0,\infty)$, we define its *lifting* $\hat{f} \in l_{L^\infty[0,\tau]}$, as follows: $\hat{f}$ is an $L^\infty[0,\tau]$-valued sequence, we denote it by $\{\hat{f}_i\}$, and for each i

$$\hat{f}_i(t) := f(t + \tau i) \qquad 0 \le t \le \tau.$$

The lifting can be visualized as taking a continuous-time signal and breaking it up into a sequence of "pieces" each corresponding to the function over an interval of length τ (see Fig. 2). Let us denote this lifting by W_τ: $L_e^\infty[0,\infty) \to l_{L^\infty[0,\tau]}$. W_τ is a linear isomorphism, furthermore, if restricted to $L^\infty(0,\infty)$, then W_τ: $L^\infty[0,\infty) \to l^\infty_{L^\infty[0,\tau]}$ is an isometry, i.e., it preserves norms.

Using the lifting of signals, one can define a lifting on systems. Let G be a linear continuous-time system on $L_e^\infty[0,\infty)$, then its *lifting* $\hat{G}$ is the discrete-time system $\hat{G} := W_\tau G W_\tau^{-1}$, this is illustrated in the commutative diagram below:

$$\begin{tikzcd} l_{L^\infty[0,\tau]} \arrow[r, "\hat{G}"] \arrow[d, "W_\tau^{-1}"'] & l_{L^\infty[0,\tau]} \\ L_e^\infty[0,\infty) \arrow[r, "G"] & L_e^\infty[0,\infty) \arrow[u, "W_\tau"'] \end{tikzcd}$$

Thus, $\hat{G}$ is a system that operates on Banach space ($L^\infty[0,\tau]$) valued signals, we will call such systems infinite dimensional. Note that since W_τ is an isometry, if G is stable, i.e., a bounded linear map on L^∞ then $\hat{G}$ is also stable, and furthermore, their respective induced norms are equal, $\|\hat{G}\| = \|G\|$. The correspondence between a system and its lifting also preserves algebraic system properties such as addition, cascade decomposition and feedback (see [1] for details).

The usefulness of the lifting in the sampled-data problem is the fact that if G is a τ-periodic system, then $\hat{G}$ commutes with the shift on $l_{L^\infty[0,\tau]}$, that is, $\hat{G}$ is shift invariant. This basic fact allows us to treat continuous-time periodic systems as discrete-time time-invariant systems, albeit infinite-dimensional systems.

State space models can be found for the lifted systems. To illustrate, let G be a continuous-time time-invariant system given by a state space realization $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. In [1] it was shown that the lifting $\hat{G}$ has a state space

[1] Essentially the same technique was arrived at independently in [22] and [23].

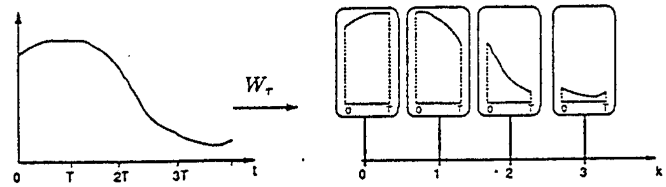

Fig. 2. W_τ: $L_e^p[0,\infty) \to l_{L^p[0,\tau]}$.

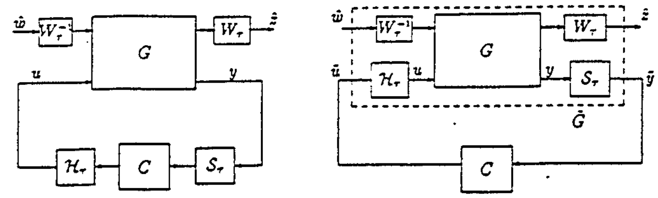

Fig. 3. Equivalent problem.

realization given by:

$$\hat{G} = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} e^{A\tau} & e^{A(\tau-\hat{s})}B \\ \hline Ce^{A\hat{t}} & Ce^{A(\hat{t}-\hat{s})}\mathbf{1}_{(\hat{t}-\hat{s})}B + D\delta(\hat{t}-\hat{s}) \end{array}\right]$$

$$\begin{aligned} \hat{B}&: \quad L^\infty[0,\tau] \to \mathbb{R}^{n_x} \\ \hat{A}&: \quad \mathbb{R}^{n_x} \to \mathbb{R}^{n_x} \\ \hat{C}&: \quad \mathbb{R}^{n_x} \to L^\infty[0,\tau] \\ \hat{D}&: \quad L^\infty[0,\tau] \to L^\infty[0,\tau] \end{aligned} \tag{1}$$

where the operators $\hat{C}, \hat{B}, \hat{D}$ are given in terms of their kernel functions, and $\mathbf{1}_{(\cdot)}$ is the unit step function.

Notation: It simplifies the notation greatly to use the same symbol for an operator and its kernel, for example, $\hat{D}(t,s)$ [or $\hat{B}(s)$] refer to the kernel functions representing the operator $\hat{D}$ (or $\hat{B}$). For operators that map a function space to $\mathbb{R}^n$, such as $\hat{B}$ above, we generally use s (or $\hat{s}$) to denote the variable of the kernel function, and for operators that map $\mathbb{R}^n$ to a function space such as $\hat{C}$ above, we use the variable t (or $\hat{t}$). The kernel representation for the operators $\hat{B}, \hat{C}, \hat{D}$ means that their action is given by

$$\hat{B}u = \int_0^\tau \hat{B}(\hat{s})u(\hat{s})\,d\hat{s} \quad (\hat{C}x)(\hat{t}) = \hat{C}(\hat{t})x, \quad \hat{t} \in [0,\tau]$$

$$(\hat{D}u)(\hat{t}) = \int_0^\tau \hat{D}(\hat{t},\hat{s})u(\hat{s})\,d\hat{s}.$$

Note that the state space of $\hat{G}$ is finite dimensional (the n_x in $\mathbb{R}^{n_x}$ refers to the dimension of the state space of G), while its input and output spaces are infinite dimensional. This fact is significant in that, although lifted systems have infinite-dimensional input and output spaces, they can be realized with a state space of dimension no larger than the dimension of the original continuous-time state space model.

To apply the lifting to the sampled-data problem, consider again the standard problem of Fig. 1, and denote the closed-loop operator by $\mathcal{F}(G, \mathcal{H}_\tau C \mathcal{S}_\tau)$. Since the lifting is an isometry, we have that $\|\mathcal{F}(G,\mathcal{H}_\tau C\mathcal{S}_\tau)\| = \|W_\tau \mathcal{F}(G,\mathcal{H}_\tau C\mathcal{S}_\tau)W_\tau^{-1}\|$, this is shown in Fig. 3(a). In Fig. 3(b), we lump the lifting operators W_τ and W_τ^{-1} and the sample and hold operators and consider a new generalized plant $\tilde{G}$. $\tilde{G}$ is a discrete-time system with one infinite-dimensional input and output (corresponding to $\hat{w}$ and $\hat{z}$) and one finite-dimensional input and output (corresponding to $\tilde{u}$ and $\tilde{y}$). Thus, $\mathcal{F}(\tilde{G}, C) = W_\tau \mathcal{F}(G,\mathcal{H}_\tau C\mathcal{S}_\tau)W_\tau^{-1}$, which means that the closed-loop operator $\mathcal{F}(\tilde{G},C)$ is in fact the lifting of the closed-loop operator $\mathcal{F}(G,\mathcal{H}_\tau C\mathcal{S}_\tau)$. Since the lifting W_τ is an isometry, we have then characterized the L^∞-induced norm of the hybrid system as the $l^\infty_{L^\infty[0,\tau]}$-induced norm of the time-invariant system $\mathcal{F}(\tilde{G},C)$. The conclusion is that the problem of minimizing the L^∞ induced norm of the sampled-data system, is equivalent to that of minimizing the induced norm of the infinite dimensional but time-invariant system $\mathcal{F}(\tilde{G},C)$. The previous discussion together with the characterization of internal stability for hybrid systems in [12] (conditions for nonpathological sampling) yields the following theorem.

Theorem 1: Let G and $\tilde{G}$ be as in Fig. 3, then for any finite dimensional C.

i) $\mathcal{F}(G,\mathcal{H}_\tau C\mathcal{S}_\tau)$ is internally stable if and only if $\mathcal{F}(\tilde{G},C)$ is.
ii) $\|\mathcal{F}(G,\mathcal{H}_\tau C\mathcal{S}_\tau)\| = \|\mathcal{F}(\tilde{G},C)\|$.

This reformulation of the sampled-data problem to the problem with $\tilde{G}$ has several advantages, first, the controller has no "structural constraints" on it, in contrast to the previous formulation where the controller is constrained to be a sampled-data controller, i.e., of the form $\mathcal{H}_\tau C\mathcal{S}_\tau$, second, both the controller C and the generalized plant $\tilde{G}$ are shift invariant, thus, the periodicity of the original system is "removed," and third, all parts of the system are operating over the same time set (discrete time). The price paid for these advantages is the infinite dimensionality of the input and output spaces. In this paper, we will show how one can reduce the problem to a finite-dimensional one by "approximating" the input and output spaces by finite-dimensional spaces, thus, reducing the problem to a standard finite-dimensional l^1 problem.

We now present (from [1]) a state space realization for the new generalized plant $\tilde{G}$ which will be useful in studying the problem further. Let the original continuous-time plant G be given by the following realization

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right].$$

It is assumed that the sampler is preceded with a presampling filter which is a strictly causal linear system, this is a realistic assumption since an ideal sampler is not a physical device, a real sampler can be modeled as an integrator with a fast time constant followed by an ideal sampler.

The system shown above represents a generalized plant with the presampling filter absorbed in it, the fact that $D_{21} = D_{22} = 0$ is due to the strict causality of the presampling filter, this also guarantees that the ideal sampler only operates on continuous signals. It can be shown ([1]) that a realization for the generalized plant $\tilde{G}$ (Fig. 3) is given by

$$\tilde{G} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \tilde{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} e^{A\tau} & e^{A(\tau-s)}B_1 & \Psi(\tau)B_2 \\ \hline C_1 e^{At} & C_1 e^{A(t-s)}1(t-s)B_1 + D_{11}\delta(t-s) & C_1\Psi(t)B_2 + D_{12} \\ C_2 & 0 & 0 \end{array}\right]$$

where $\Psi(t) := \int_0^t e^{As}\,ds$. The system G has the following input and output spaces

$$\begin{aligned} \tilde{G}_{11}&: \quad l_{L^\infty[0,\tau]} \to l_{L^\infty[0,\tau]} \\ \tilde{G}_{12}&: \quad l_{R^{n_u}} \to l_{L^\infty[0,\tau]} \\ \tilde{G}_{21}&: \quad l_{L^\infty[0,\tau]} \to l_{R^{n_y}} \\ \tilde{G}_{22}&: \quad l_{R^{n_u}} \to l_{R^{n_y}}. \end{aligned}$$

The main theme of this paper is to approximate the infinite-dimensional input and output spaces $L^\infty[0,\tau]$ by finite-dimensional spaces. Bounds on the approximation of the closed-loop system (i.e., with controller) will be obtained that are characterized only in terms of the operators $\hat{B}_1, \hat{C}_1, \hat{D}_{12}, \hat{D}_{11}$, which in turn are characterized by the original continuous-time plant and independent of the controller.

The interpretation that can be given to the operators $\hat{B}_1, \hat{C}_1, \hat{D}_{12}, \hat{D}_{11}$ is that they characterize the inter-sample behavior of the overall system. In the lifted formulation of the sampled-data problem, the state of the system is the state of the plant $\tilde{G}$ and the state of the controller C, both of which evolve in discrete time. The controller thus has an effect on the state of the system only at the sampling instants, and the inter-sample behavior is governed only by the plant dynamics. This fact is made intuitive by the observation that in between the samples, the system is essentially operating in open loop since there is no feedback (u is constant in between samples).

The lifting of the sampled-data problem makes clear that the inter-sample dynamics are characterized by the operators $\hat{B}_1, \hat{C}_1, \hat{D}_{12}, \hat{D}_{11}$, and thus the issue of approximating these dynamics essentially amounts to approximating the operators, which are independent of the controller. The foregoing ideas are pursued in the next sections.

III. Solution Procedure

Using the lifting we are able to convert the problem of finding a controller to minimize the L^∞ induced norm of the hybrid system (Fig. 1) into the following standard problem with an infinite-dimensional generalized plant $\tilde{G}$:

$$\begin{aligned} \gamma_{\text{opt}} &:= \inf_{C \text{ stabilizing}} \|\mathscr{F}(G, \mathscr{H}_\tau C \mathscr{S}_\tau)\| \\ &= \inf_{C \text{ stabilizing}} \|\mathscr{F}(\tilde{G}, C)\|. \end{aligned} \tag{2}$$

We also note that because of Theorem 1, suboptimal solutions to the above problem will also be suboptimal (with the same norm) for the hybrid system.

The above infinite-dimensional problem is solved by an approximation procedure through solving a standard MIMO l^1 problem. The idea we use is similar to that in [10] and [14] where multirate sampling is used to obtain discrete-time systems that approximate the continuous-time behavior of hybrid systems. This approximation procedure was used in [10] to address the l^1 sampled-data problem. The approximation procedure we use is essentially equivalent to that in [10], however, since we introduce it directly as an approximation to the lifted problem (2), the nature of the approximation is more transparent and we are able to explicitly isolate the parts of the system that need to be approximated independently of the controller. The consequence is that we are able to obtain explicit bounds on the degree of approximation in terms of constants that can be computed *a priori*, and that are dependent only on the plant.

We now describe the approximation procedure. Let $\mathscr{H}_n$ and $\mathscr{S}_n$ be the following operators defined between $L_q^\infty[0,\tau]$ and $l_q^\infty(n)$($l_q^\infty(n)$ is $\mathbb{R}^{n\times q}$ with the maximum norm

$$\mathscr{S}_n: \quad L_q^\infty[0,\tau] \to l_q^\infty(n) \quad (\mathscr{S}_n u)(i) = u\left(\frac{\tau}{n}i\right); \qquad u \in L_q^\infty[0,\tau]$$

$$\mathscr{H}_n: \quad l_q^\infty(n) \to L_q^\infty[0,\tau] \quad (\mathscr{H}_n u)(t) = u\left(\left\lfloor \frac{tn}{\tau} \right\rfloor\right); \qquad \{u(i)\} \in l_q^\infty(n)$$

(strictly speaking, $\mathscr{S}_n$ is not an operator on L_q^∞ but on the subspace of left and right continuous functions, this distinction is irrelevant here since in our setting, assumptions are made to guarantee that $\mathscr{S}_n$ operates only on continuous signals), the above operators can be thought of as "fast" sample and hold operators (see Fig. 5). For simplicity of notation we will suppress the dimension q in the sequel.

Now to approximate the infinite-dimensional problem, we use the approximate closed-loop system $\mathscr{S}_n\mathscr{F}(\tilde{G}, C)\mathscr{H}_n$ (see Fig. 4), and for each n we define

$$\gamma_n := \inf_{C \text{ stabilizing}} \|\mathscr{S}_n\mathscr{F}(\tilde{G}, C)\mathscr{H}_n\|. \tag{3}$$

This new problem now involves the induced norm over

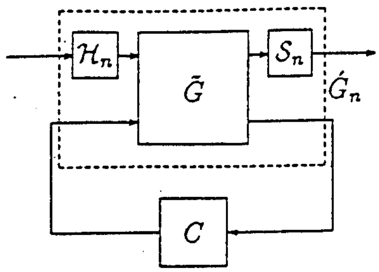

Fig. 4. The system $\acute{G}_n$.

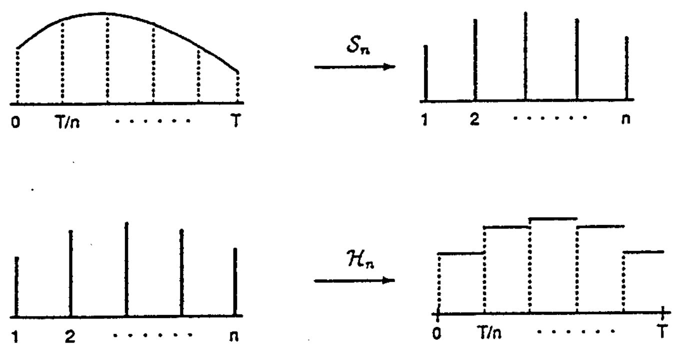

Fig. 5. The operators $\mathcal{S}_n$ and $\mathcal{H}_n$.

$l^\infty_{l^\infty(n)}$, i.e., it is a standard MIMO l^1 problem.

Let us denote the generalized plant associated with $\mathcal{S}_n\mathcal{F}(\bar{G},C)\mathcal{H}_n$ by $\acute{G}_n$, that is, $\acute{G}_n$ is such that (see Fig. 4)

$$\mathcal{S}_n\mathcal{F}(\bar{G},C)\mathcal{H}_n = \mathcal{F}(\acute{G}_n, C).$$

A realization for $\acute{G}_n$ is given by,

$$\acute{G}_n = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1\mathcal{H}_n & \bar{B}_2 \\ \hline \mathcal{S}_n\hat{C}_1 & \mathcal{S}_n\hat{D}_{11}\mathcal{H}_n & \mathcal{S}_n\bar{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right] =: \left[\begin{array}{c|cc} \hat{A} & \acute{B}_1 & \bar{B}_2 \\ \hline \acute{C}_1 & \acute{D}_{11} & \acute{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right].$$

The new operators, which are now matrices, are computed to be

$$\acute{C}_1 = \begin{bmatrix} C_1 \\ C_1 e^{A\tau/n} \\ \vdots \\ C_1(e^{A\tau/n})^{n-1} \end{bmatrix}$$

$$\acute{D}_{12} = \begin{bmatrix} D_{12} \\ C_1\Psi(\tau/n)B_2 + D_{12} \\ \vdots \\ C_1\Psi\left(\dfrac{\tau(n-1)}{n}\right)B_2 + D_{12} \end{bmatrix},$$

$$\acute{D}_{11} = \left\{\left[\begin{array}{c|c} e^{A\tau/n} & \Psi(\tau/n)B_1 \\ \hline C_1 & D_{11} \end{array}\right]\right\}_n$$

$$\acute{B}_1 = \Psi(\tau/n)\begin{bmatrix} B_1 & e^{A\tau/n}B_1 & \cdots & (e^{A\tau/n})^{n-1} & B_1 \end{bmatrix}$$

where $\{\cdot\}_n$ means the first $n \times n$ blocks of the impulse response matrix of the discrete-time system given by the realization in $\{\cdot\}$.

The solution to the original infinite-dimensional problem (and thus to the sampled-data problem) is as follows: n can be chosen large enough such that if the designed controller C_n is almost optimal for the approximate problem (3), then it is almost optimal for the original problem (2). In essence, this approximation scheme "converges," i.e., one can obtain almost optimal controllers by choosing n large enough and solving a MIMO l^1 problem. Exactly what convergence means here is described next.

IV. Design Bounds

In this section we investigate the nature of the approximation of $\|\mathcal{F}(\bar{G},C)\|$ by $\|\mathcal{F}(\acute{G}_n,C)\|$. In order to show that the synthesis procedure outlined in the previous section yields controllers with performance arbitrarily close to the optimal, one needs to obtain explicit bounds on the degree of approximation of $\|\mathcal{F}(\bar{G},C)\|$ by $\|\mathcal{F}(\acute{G}_n,C)\|$.

Let us begin with analysis. Note that since $\|\mathcal{F}(\bar{G},C)\|$ is an infinite-dimensional system, its $l^\infty_{L^\infty[0,\tau]}$-induced norm is not readily computable. A method of computing $\|\mathcal{F}(\bar{G},C)\|$ comes from the limit

$$\|\mathcal{F}(\bar{G},C)\| = \lim_{n\to\infty}\|\mathcal{S}_n\mathcal{F}(\bar{G},C)\mathcal{H}_n\| =: \lim_{n\to\infty}\|\mathcal{F}(\acute{G}_n,C)\| \tag{4}$$

for a fixed C. This formula can be proved using arguments about the approximation of continuous functions by simple functions in L^∞ ([19]), and also follows immediately from the main inequality below. Since $\mathcal{F}(\acute{G}_n,C)$ is a time-invariant MIMO system and $\|\mathcal{F}(\acute{G}_n,C)\|$ is its l^1 norm, it can be computed to any desired accuracy, consequently, by (4) the actual norm, $\|\mathcal{F}(\bar{G},C)\|$ can be computed to any desired accuracy. However, (4) is by far not sufficient to show the convergence of the synthesis procedure, since given only (4), the rate of convergence may depend on the choice of C.

Our objective is to obtain explicit bounds on $\|\mathcal{F}(\bar{G},C)\|$ that do not depend on the controller in the following form

Main Inequality: There are constants K_o and K_1 which depend only on G, such that for $n \geq 2n_x$, and τ/n nonpathological

$$\|\mathcal{F}(\acute{G}_n,C)\| \leq \|\mathcal{F}(\bar{G},C)\| \leq \frac{K_1}{n} + \left(1+\frac{K_o}{n}\right)\|\mathcal{F}(\acute{G}_n,C)\|. \tag{5}$$

Remarks:

a) The significance of the bound (5) is that it is exactly what is needed for synthesis. When one performs an l^1 design on the approximate discretization $\acute{G}_n$, the result is a controller that keeps $\|\mathcal{F}(\acute{G}_n,C)\|$ small, but the objective is to keep the L^∞-induced norm of the hybrid system (or equivalently $\|\mathcal{F}(\bar{G},C)\|$) small, and the inequality (5) guarantees this. It is thus essential that we bound the hybrid norm from *above* by a function of $\|\mathcal{F}(\acute{G}_n,C)\|$.

b) The above inequality shows that the approximation converges at a rate of $(1/n)$.

The first inequality in (5) is easy to obtain, first note that

$$\|\mathscr{F}(\acute{G}_n, C)\| \leq \|\mathscr{F}(\tilde{G}, C)\| \qquad \forall n,$$

since

$$\begin{aligned}\|\mathscr{F}(\acute{G}_n, C)\| &= \|\mathscr{S}_n \mathscr{F}(\tilde{G}, C)\mathscr{H}_n\| \\ &\leq \|\mathscr{S}_n\|\|\mathscr{F}(\tilde{G}, C)\|\|\mathscr{H}_n\| \leq \|\mathscr{F}(\tilde{G}, C)\|\end{aligned}$$

because $\|\mathscr{H}_n\| \leq 1$ on $l^\infty(n)$ and $\|\mathscr{S}_n\| \leq 1$ on the subspace of L^∞ for which it is defined.

One way to utilize the main inequality for getting *a priori* guarantees on the hybrid norm in terms of the discrete-time l^1 problem is guided by the following; for a fixed n, if one performs a MIMO l^1 design (as in [9], [17]) on $\acute{G}_n$ and obtains a $\gamma_n + \epsilon$ optimal controller (given by C_n), i.e., $\|\mathscr{F}(\acute{G}_n, C_n)\| \leq \gamma_n + \epsilon$, then inequality (5) provides that if C_n is implemented in the hybrid system, then

$$\begin{aligned}\gamma_{\text{opt}} \leq \|\mathscr{F}(\tilde{G}, G_n)\| &\leq \frac{K_1}{n} + \left(1 + \frac{K_o}{n}\right)\|\mathscr{F}(\acute{G}_n, C)\| \\ &\leq \frac{K_1}{n} + \left(1 + \frac{K_o}{n}\right)(\gamma_n + \epsilon) \\ &\leq \frac{K_1}{n} + \left(1 + \frac{K_o}{n}\right)(\gamma_{\text{opt}} + \epsilon) \qquad (6)\end{aligned}$$

where the last inequality follows from $\gamma_n \leq \gamma_{\text{opt}}$, which is a consequence of the first inequality in (5).

The above inequality can be simplified by using an upper bound on γ_{opt}, such a bound can be obtained by finding any stabilizing controller C_o and computing an upper bound on the hybrid norm of $\mathscr{F}(\tilde{G}, C_o)$ (by using the main inequality with a large n). Call that upper bound M. Then by using $\gamma_{\text{opt}} \leq M$, inequality (6) can be rewritten as

$$\gamma_{\text{opt}} \leq \|\mathscr{F}(\tilde{G}, C_n)\| \leq \frac{K_1 + K_o(M + \epsilon)}{n} + \epsilon + \gamma_{\text{opt}}.$$

Thus, in order that C_n guarantees $\|\mathscr{F}(G, \mathscr{H}_\tau C_n \mathscr{S}_\tau)\| \leq \gamma_{\text{opt}} + \delta$ for any $\delta > 0$, we choose ϵ and n *a priori* to satisfy

$$\delta \leq \frac{K_1 + K_o(M + \epsilon)}{n} + \epsilon.$$

It is worthwhile noting that the problem of minimizing $\|\mathscr{F}(\acute{G}_n, C)\|$ is immediately a standard l^1 problem with time-invariant plant. Also, we note that even though the approximation problem is essentially equivalent to a multirate sampled-data problem, it reflects no structural constraints on the controller. General multirate sampled problems do not share this property (see [7]).

The next section is devoted to the derivation of the main inequality (5). Several interesting issues come up, and we get bounds on the approximation by characterizing the approximation of the infinite-dimensional parts of $\tilde{G}$, namely the operators $\hat{B}_1, \hat{C}_1, \tilde{D}_{12}, \hat{D}_{11}$.

V. Decomposition and Approximation of $\tilde{G}$

It will be very helpful in the derivation of (5) to introduce a decomposition of the infinite-dimensional system $\tilde{G}$ by "extracting" the infinite-dimensional parts of the system. The basic idea is roughly that the behavior of the hybrid system between samples is essentially governed by the infinite-dimensional parts of $\tilde{G}$, namely the operators $\hat{B}_1$, $\hat{C}_1$, $\tilde{D}_{12}$, and $\hat{D}_{11}$. These operators are independent of the controller, and thus it should be possible to approximate the behavior in between the samples independently of the controller by "approximating" the aforementioned operators. To illustrate this point further, we first decompose $\tilde{G}$ as

$$\tilde{G} = \tilde{G}_o + \begin{bmatrix} \hat{D}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{G}_o := \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \tilde{B}_2 \\ \hline \hat{C}_1 & 0 & \tilde{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right]$$

and we note that $\tilde{G}_o$ can be further decomposed as

$$\tilde{G}_0 = \begin{bmatrix} \begin{bmatrix} \hat{C}_1 & \tilde{D}_{12} \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c|cc} \hat{A} & I & \tilde{B}_2 \\ \hline \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \tilde{C}_2 & 0 & 0 \end{array}\right] \begin{bmatrix} \hat{B}_1 & 0 \\ 0 & I \end{bmatrix}. \qquad (7)$$

This decomposition is illustrated in Fig. 6. The closed-loop mapping $\mathscr{F}(\tilde{G}, C)$ is correspondingly decomposed as

$$\begin{aligned}\mathscr{F}(\tilde{G}, C) &= \hat{D}_{11} + \mathscr{F}(\tilde{G}_o, C) \\ &= \hat{D}_{11} + \begin{bmatrix} \hat{C}_1 & \tilde{D}_{12} \end{bmatrix}\mathscr{F}(\tilde{G}_{oo}, C)\hat{B}_1. \qquad (8)\end{aligned}$$

We will use the notation $\hat{\mathscr{C}} := [\hat{C}_1 \ \tilde{D}_{12}]$, and call $\hat{\mathscr{C}}$ the output operator and $\hat{B}_1$ the input operator.

With this decomposition, $\tilde{G}_{oo}$ is finite dimensional, and $\hat{\mathscr{C}}, \hat{B}_1$ are finite rank operators

$$\hat{\mathscr{C}}: \mathbb{R}^{n_x + n_u} \to L^\infty[0, \tau], \qquad \hat{B}_1: L^\infty[0, \tau] \to \mathbb{R}^{n_x}.$$

As (8) shows, only a finite-dimensional part of the system [i.e., $\mathscr{F}(G(\tilde{G}_{oo}, C)$] is dependent on the controller, while the infinite-dimensional parts are independent of C. Roughly speaking, the controller (being discrete time) only effects the hybrid system at the sampling instants, while in between the samples, the systems evolution is governed by the operators $\hat{D}_{11}, \hat{\mathscr{C}}, \hat{B}_1$, which are in turn dependent only on the dynamics of the original generalized plant G.

The remainder of this section and the appendixes are devoted to deriving the main inequality, and can be skipped without loss of continuity.

We now consider the issue of "approximating" the infinite-dimensional plant $\tilde{G}$ by a finite-dimensional plant $\acute{G}_n$. First we note that the two norms to be compared are

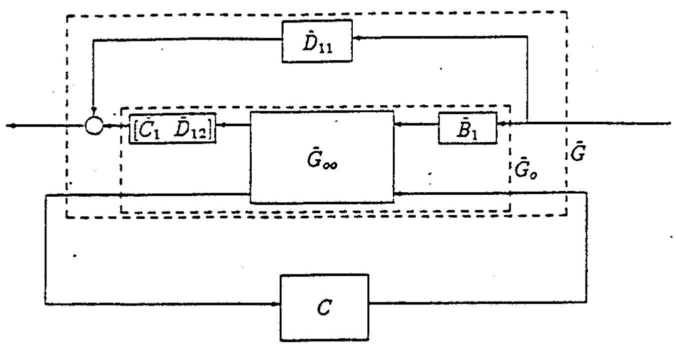

Fig. 6. Decomposition of $\tilde{G}$.

of $\mathcal{F}(\tilde{G}, C)$, which has $L^\infty[0,\tau]$ as an input–output space, and of $\mathcal{S}_n\mathcal{F}(\tilde{G}, C)\mathcal{H}_n$, which has $l^\infty(n)$ as an input–output space. Therefore, it is not strictly true that $\mathcal{S}_n\mathcal{F}(\tilde{G}, C)\mathcal{H}_n$ approximates $\mathcal{F}(\tilde{G}, C)$ since comparisons like $\|\mathcal{F}(\tilde{G}, C) - \mathcal{S}_n\mathcal{F}(\tilde{G}, C)\mathcal{H}_n\| \le \epsilon$ do not make sense. We will replace $\mathcal{S}_n\mathcal{F}(\tilde{G}, C)\mathcal{H}_n$ by another system which has the same norm, but truly approximates $\mathcal{F}(\tilde{G}, C)$.

Define the following operator (the normalized integration operator) $\mathcal{J}_n$: $L^\infty[0,\tau] \to l^\infty(n)$ by

$$(\mathcal{J}_n(u))(i) := \frac{n}{\tau}\int_{i\tau/n}^{(i+1)\tau/n} u(t)\, dt.$$

The following properties of $\mathcal{J}_n$ can be easily checked: $\mathcal{J}_n$ is a linear operator, $\|\mathcal{J}_n\| = 1$, and $\mathcal{J}_n$ is a left inverse to $\mathcal{H}_n$, i.e. $\mathcal{J}_n\mathcal{H}_n$ = identity. If $\mathcal{J}_n$ is regarded as an operator on $L^1[0,\tau]$, i.e., $\mathcal{J}_n$: $L^1[0,\tau] \to l^1(n)$, then it is easily shown that $\mathcal{H}_n$ is the adjoint of $(\tau/n)\mathcal{J}_n$, that is $((\tau/n)\mathcal{J}_n)^* = \mathcal{H}_n$. Similarly, if $\mathcal{H}_n$ is regarded as an operator on $l^1(n)$, i.e., $\mathcal{H}_n$: $l^1(n) \to L^1[0,\tau]$, then $\mathcal{H}_n^* = (\tau/n)\mathcal{J}_n$, which also implies that $(\mathcal{H}_n\mathcal{J}_n)^* = \mathcal{H}_n\mathcal{J}_n$.

Let us denote by $T := \mathcal{F}(\tilde{G}, C)$, and by $\acute{T}_n := \mathcal{F}(\acute{G}_n, C)$. As already mentioned, T and $\acute{T}_n$ cannot be compared directly since they do not have the same input and output space. The operator $\mathcal{J}_n$ will allow us to form a system $\overline{T}_n$ with norm equal to that of $\acute{T}_n$, but with the same input and output spaces as T.

Lemma 2: Define the system $\overline{T}_n := (\mathcal{H}_n\mathcal{S}_n)T(\mathcal{H}_n\mathcal{J}_n)$, then

$$\|\overline{T}_n\| = \|\acute{T}_n\|.$$

Proof: It is true that $\|\mathcal{S}_nT\mathcal{H}_n\mathcal{J}_n\| = \|\mathcal{S}_nT\mathcal{H}_n\|$ since

$$\|\mathcal{S}_nT\mathcal{H}_n\mathcal{J}_n\| \le \|\mathcal{S}_nT\mathcal{H}_n\|\,\|\mathcal{J}_n\| \le \|\mathcal{S}_nT\mathcal{H}_n\|,$$

and

$$\|\mathcal{S}_nT\mathcal{H}_n\| \le \|\mathcal{S}_nT\mathcal{H}_n\mathcal{J}_n\mathcal{H}_n\| \le \|\mathcal{S}_nT\mathcal{H}_n\mathcal{J}_n\|.$$

Also, since $\mathcal{H}_n$: $l^\infty(n) \to L^\infty[0,\tau]$ is an isometry, we conclude that

$$\|\overline{T}_n\| := \|\mathcal{H}_n\mathcal{S}_nT\mathcal{H}_n\mathcal{J}_n\| = \|\mathcal{S}_nT\mathcal{H}_n\mathcal{J}_n\| = \|\mathcal{S}_nT\mathcal{H}_n\| =: \|\acute{T}_n\|.$$

∎

Remark: The above lemma is of general interest since it provides a systematic way of addressing the question of how a discretized system $\mathcal{S}_\tau H\mathcal{H}_\tau$ "approximates" the original system H, by comparing the systems H and $\overline{H} := (\mathcal{H}_\tau\mathcal{S}_\tau)H(\mathcal{H}_\tau\mathcal{J}_\tau)$. This comparison is typically easier since H and $\overline{H}$ are both continuous-time systems with the same input and output spaces.

Let $\overline{G}_n$ be the generalized plant corresponding to the closed-loop operator $\overline{T}_n$, i.e., $\overline{T}_n = \mathcal{F}(\overline{G}_n, C)$. $\overline{G}_n$ is defined by

$$\overline{G}_n = \begin{bmatrix} \mathcal{H}_n\mathcal{S}_n & 0 \\ 0 & I \end{bmatrix} \tilde{G} \begin{bmatrix} \mathcal{H}_n\mathcal{J}_n & 0 \\ 0 & I \end{bmatrix}.$$

The consequence of Lemma 2 is that one only needs to show inequality (5) with $\mathcal{F}(\overline{G}_n, C)$ instead of $\mathcal{F}(\acute{G}_n, C)$. As already mentioned, the advantage is that $\mathcal{F}(\overline{G}_n, C)$ has the same input and output spaces as $\mathcal{F}(\tilde{G}, C)$, namely $L^\infty[0,\tau]$.

Next, we will show that $\mathcal{F}(\overline{G}_n, C)$ actually approximates $\mathcal{F}(\tilde{G}, C)$, and this will yield the main inequality (5).

Approximation of $\tilde{G}$: The approximation of $\tilde{G}$ will be done in two parts corresponding to the decomposition $\mathcal{F}(\tilde{G}, C) = \hat{D}_{11} + \mathcal{F}(\tilde{G}_o, C) = \hat{D}_{11} + \hat{\mathcal{O}}\mathcal{F}(\tilde{G}_{oo}, C)\hat{B}_1$. It will be useful in this section to use a short hand notation for (see Fig. 7)

$$T_o := \hat{\mathcal{O}}\mathcal{F}(\tilde{G}_{oo}, C)\hat{B}_1 \qquad T_{oo} := \mathcal{F}(\tilde{G}_{oo}, C) \tag{9}$$

$$\overline{T}_{on} := (\mathcal{H}_n\mathcal{S}_n)T_o(\mathcal{H}_n\mathcal{J}_n) \qquad \overline{D}_n := (\mathcal{H}_n\mathcal{S}_n)\hat{D}_{11}(\mathcal{H}_n\mathcal{J}_n) \tag{10}$$

and corresponding to the decomposition $T = \hat{D}_{11} + T_{on}$, we have

$$\overline{T}_n = (\mathcal{H}_n\mathcal{S}_n)\big(\hat{D}_{11} + T_o\big)(\mathcal{H}_n\mathcal{J}_n) = \overline{D}_n + \overline{T}_{on}.$$

We will first show that $\overline{T}_{on}$ approximates T_o, then we show that $\overline{D}_n$ approximates $\hat{D}_{11}$.

Proposition 3: Let $n \ge 2n_x$, such that τ/n is not a pathological sampling period, there exists a constant K_o which depends only on G, such that

$$\|T_o - \overline{T}_{on}\| \le \frac{K_o}{n}\|\overline{T}_{on}\|.$$

Remark: It is important that the above bound is in terms of $\|\overline{T}_{on}\|$ which corresponds to part of $\mathcal{F}(\acute{G}_n, C)$. The reason being that in the main inequality, we must bound the norm of the hybrid system from *above* by the norm of the discretized system $\mathcal{F}(\acute{G}_n, C)$. In fact, it is much easier to produce an inequality as above but with $\|T_o\|$ on the right-hand side, but this would not be useful for bounding the norm of the hybrid system.

Proof: The proof makes use of the decomposition of $T_o = \hat{\mathcal{O}}T_{oo}\hat{B}_1$, and of its approximation $\overline{T}_{on} = (\mathcal{H}_n\mathcal{S}_n)\hat{\mathcal{O}}T_{oo}\hat{B}_1(\mathcal{H}_n\mathcal{J}_n)$. The basic idea of the proof (on the output side) is that $(\mathcal{H}_n\mathcal{S}_n)$ operates on functions in $\mathcal{R}_{(\hat{\mathcal{O}})} \subset L^\infty[0,\tau]$, and functions in $\mathcal{R}_{(\hat{\mathcal{O}})}$ are continuous and there are bounds on their rate of change (depending on the dynamics of the plant), so on $\mathcal{R}_{(\hat{\mathcal{O}})}$ the operator $(\mathcal{H}_n\mathcal{S}_n)$ approximates the identity, and it also has a left inverse which approximates the identity as $n \to \infty$.

We now approximate from the output side. Lemma 4 below states that $(\mathcal{H}_n\mathcal{S}_n)$ has a left inverse on $\mathcal{R}_{(\hat{\mathcal{O}})}$, i.e., there exists $(\mathcal{H}_n\mathcal{S}_n)^{-L}$: $\mathcal{R}_{(\mathcal{H}_n\mathcal{S}_n\hat{\mathcal{O}})} \to \mathcal{R}_{(\hat{\mathcal{O}})} \subset L^\infty[0,\tau]$

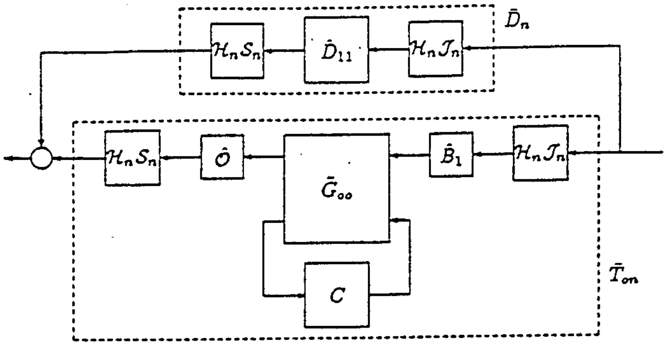

Fig. 7. Decomposition of the approximate system $\mathscr{S}(\bar{G}_n, C)$.

such that $(\mathscr{H}_n\mathscr{S}_n)^{-L}(\mathscr{H}_n\mathscr{S}_n) =$ identity on $\mathscr{R}_{(\hat{\mathscr{O}})}$. We now establish

$$\begin{aligned}&\|(\mathscr{H}_n\mathscr{S}_n)T_o - T_o\| \\ &\quad = \left\|(\mathscr{H}_n\mathscr{S}_n)\hat{\mathscr{O}}T_{oo}\hat{B}_1 - \left((\mathscr{H}_n\mathscr{S}_n)^{-L}(\mathscr{H}_n\mathscr{S}_n)\hat{\mathscr{O}}T_{oo}\hat{B}_1\right)\right\| \\ &\quad \le \left\|\left(I - (\mathscr{H}_n\mathscr{S}_n)^{-L}\right)\big|_{\mathscr{R}(\mathscr{H}_n\mathscr{S}_n\hat{\mathscr{O}})}\right\| \, \|(\mathscr{H}_n\mathscr{S}_n)T_o\|\end{aligned}$$

where the operator I is the identity, or the embedding I: $\mathscr{R}_{(\mathscr{H}_n\mathscr{S}_n\hat{\mathscr{O}})} \to L^\infty[0, \tau]$. Also from Lemma 4, we have that $\|(I - (\mathscr{H}_n\mathscr{S}_n)^{-L})|_{\mathscr{R}(\mathscr{H}_n\mathscr{S}_n\hat{\mathscr{O}})}\| \le (K_{\hat{\mathscr{O}}}/n)$, this implies

$$\|(\mathscr{H}_n\mathscr{S}_n)T_o - T_o\| \le \frac{K_{\hat{\mathscr{O}}}}{n}\|(\mathscr{H}_n\mathscr{S}_n)T_o\|. \tag{11}$$

Now, to approximate on the input side, we need to take preadjoints (see Appendix B):

$$\begin{aligned}&\|(\mathscr{H}_n\mathscr{S}_n)T_o - (\mathscr{H}_n\mathscr{S}_n)T_o(\mathscr{H}_n\mathscr{T}_n)\| \\ &\quad = \|(\mathscr{H}_n\mathscr{S}_n)\hat{\mathscr{O}}T_{oo}\hat{B}_1 - (\mathscr{H}_n\mathscr{S}_n)\hat{\mathscr{O}}T_{oo}\hat{B}_1(\mathscr{H}_n\mathscr{T}_n)\| \\ &\quad = \left\|(\mathscr{H}_n\mathscr{S}_n)\hat{\mathscr{O}}T_{oo}\left(\hat{B}_1 - \hat{B}_1(\mathscr{H}_n\mathscr{T}_n)\right)\right\| \\ &\quad = \left\|\left({}^*\hat{B}_1 - {}^*\mathscr{T}_n^*\mathscr{H}_n^*\hat{B}_1\right)^*\left((\mathscr{H}_n\mathscr{S}_n)\hat{\mathscr{O}}T_{oo}\right)\right\| \\ &\quad = \left\|\left({}^*\hat{B}_1 - (\mathscr{H}_n\mathscr{T}_n)^*\hat{B}_1\right)^*\left((\mathscr{H}_n\mathscr{S}_n)\hat{\mathscr{O}}T_{oo}\right)\right\|.\end{aligned}$$

From Lemma 4 below, $(\mathscr{H}_n\mathscr{T}_n)$ has a left inverse when restricted to $\mathscr{R}_{({}^*\hat{B}_1)}$, i.e., $(\mathscr{H}_n\mathscr{T}_n)^{-L}$ is such that $(\mathscr{H}_n\mathscr{T}_n)^{-L}(\mathscr{H}_n\mathscr{T}_n) =$ identity on $\mathscr{R}_{({}^*\hat{B}_1)} \subset L^1[0, \tau]$, therefore

$$\begin{aligned}&\|(\mathscr{H}_n\mathscr{S}_n)T_o - (\mathscr{H}_n\mathscr{S}_n)T_o(\mathscr{H}_n\mathscr{T}_n)\| \\ &\quad = \left\|\left((\mathscr{H}_n\mathscr{T}_n)^{-L}(\mathscr{H}_n\mathscr{T}_n)^*\hat{B}_1\right.\right. \\ &\qquad \left.\left. - (\mathscr{H}_n\mathscr{T}_n)^*\hat{B}_1\right)^*((\mathscr{H}_n\mathscr{S}_n)\mathscr{O}T_{oo})\right\| \\ &\quad = \left\|\left((\mathscr{H}_n\mathscr{T}_n)^{-L} - I\right)\big|_{\mathscr{R}(\mathscr{H}_n\mathscr{T}_n{}^*\hat{B}_1)}{}^* \right. \\ &\qquad \left. \cdot\left((\mathscr{H}_n\mathscr{S}_n)\hat{\mathscr{O}}T_{oo}\hat{B}_1(\mathscr{H}_n\mathscr{T}_n)\right)\right\| \\ &\quad \le \left\|\left((\mathscr{H}_n\mathscr{T}_n)^{-L} - I\right)\big|_{\mathscr{R}(\mathscr{H}_n\mathscr{T}_n{}^*\hat{B}_1)}\right\| \, \|(\mathscr{H}_n\mathscr{S}_n)T_o(\mathscr{H}_n\mathscr{T}_n)\| \\ &\quad \le \frac{K_{\hat{B}}}{n}\|\bar{T}_{on}\|\end{aligned} \tag{12}$$

where the last step is again from Lemma 4.

Combining inequalities (11) and (12), we get

$$\begin{aligned}\|T_o - \bar{T}_{on}\| &= \|T_o - (\mathscr{H}_n\mathscr{S}_n)T_o(\mathscr{H}_n\mathscr{T}_n)\| \\ &= \|T_o - (\mathscr{H}_n\mathscr{S}_n)T_o + (\mathscr{H}_n\mathscr{S}_n)T_o \\ &\quad - (\mathscr{H}_n\mathscr{S}_n)T_o(\mathscr{H}_n\mathscr{T}_n)\| \\ &\le \|T_o - (\mathscr{H}_n\mathscr{S}_n)T_o\| + \|(\mathscr{H}_n\mathscr{S}_n)T_o \\ &\quad - (\mathscr{H}_n\mathscr{S}_n)T_o(\mathscr{H}_n\mathscr{T}_n)\| \\ &\le \frac{K_{\hat{\mathscr{O}}}}{n}\|(\mathscr{H}_n\mathscr{S}_n)T_o\| + \frac{K_{\hat{B}}}{n}\|\bar{T}_{on}\|\end{aligned}$$

but (12) also implies that $\|(\mathscr{H}_n\mathscr{S}_n)T_o\| \le (1 + (K_{\hat{B}}/n))\|\bar{T}_{on}\|$, therefore

$$\|T_o - \bar{T}_{on}\| \le \left[\frac{K_{\hat{\mathscr{O}}}}{n}\left(1 + \frac{K_{\hat{B}}}{n}\right) + \frac{K_{\hat{B}}}{n}\right]\|\bar{T}_{on}\| \le \frac{K_o}{n}\|\bar{T}_{on}\|,$$

where $K_o := K_{\hat{\mathscr{O}}} + K_{\hat{\mathscr{O}}}K_{\hat{B}} + K_{\hat{B}}$. ■

Lemma 4 below captures the idea that $(\mathscr{H}_n\mathscr{S}_n)\hat{\mathscr{O}}$ approximates $\hat{\mathscr{O}}$, because the sampling operator $\mathscr{S}_n$ samples only elements in $\mathscr{R}(\hat{\mathscr{O}})$, and since there is a bound on the variation of functions in $\mathscr{R}(\hat{\mathscr{O}})$, one can get a bound on how well $(\mathscr{H}_n\mathscr{S}_n)$ approximates elements in $\mathscr{R}(\hat{\mathscr{O}})$. Similar arguments are made about $(\mathscr{H}_n\mathscr{T}_n)^*\hat{B}_1$. This lemma is the key to obtaining approximations that are independent of the controllers, since the behavior of the signals in the input and output spaces is governed by $\hat{\mathscr{O}}$ and $\hat{B}_1$, the nature of the approximation depends on these two operators and not C. The rate of convergence of the approximations is determined by the constants $K_{\hat{B}}$, $K_{\hat{\mathscr{O}}}$, which are completely determined by the operators $\hat{B}$ and $\hat{\mathscr{O}}$, respectively, which in turn, are completely determined by the original plant.

Lemma 4: Assume $n \ge 2n_x$, and τ/n is not a pathological sampling period, then

a) $\exists$ an operator $(\mathscr{H}_n,\mathscr{T}_n)^{-L}$: $\mathscr{R}_{(\mathscr{H}_n\mathscr{T}_n{}^*\hat{B}_1)} \to L^1[0, \tau]$ such that $(\mathscr{H}_n\mathscr{T}_n)^{-L}(\mathscr{H}_n\mathscr{T}_n)|_{\mathscr{R}({}^*\hat{B}_1)} =$ identity,

$$\begin{array}{ccccc} L^1[0,\tau] & & L^1[0,\tau] & & L^1[0,\tau] \\ \cup & (\mathscr{H}_n\mathscr{T}_n)^{-L} & \cup & (\mathscr{H}_n\mathscr{T}_n) & \cup \\ \mathscr{R}_{({}^*\hat{B}_1)} & \leftarrow & \mathscr{R}_{(\mathscr{H}_n\mathscr{T}_n{}^*\hat{B}_1)} & \leftarrow & \mathscr{R}_{({}^*\hat{B}_1)} \end{array}$$

and a constant $K_{\hat{B}}$, such that

$$\left\|\left(I - (\mathscr{H}_n\mathscr{T}_n)^{-L}\right)\big|_{\mathscr{R}(\mathscr{H}_n\mathscr{T}_n{}^*\hat{B}_1)}\right\| \le \frac{K_{\hat{B}}}{n}.$$

b) $\exists$ an operator $(\mathscr{H}_n\mathscr{S}_n)^{-L}$: $\mathscr{R}_{(\mathscr{H}_n\mathscr{S}_n\hat{\mathscr{O}})} \to L^\infty[0, \tau]$ such that $(\mathscr{H}_n\mathscr{S}_n)^{-L}(\mathscr{H}_n\mathscr{S}_n)\|_{\mathscr{R}(\hat{\mathscr{O}})} =$ identity,

$$\begin{array}{ccccc} L^\infty[0,\tau] & & L^\infty[0,\tau] & & L^\infty[0,\tau] \\ \cup & (\mathscr{H}_n\mathscr{S}_n)^{-L} & \cup & (\mathscr{H}_n\mathscr{S}_n) & \cup \\ \mathscr{R}_{(\hat{\mathscr{O}})} & \leftarrow & \mathscr{R}_{(\mathscr{H}_n\mathscr{S}_n\hat{\mathscr{O}})} & \leftarrow & \mathscr{R}_{(\hat{\mathscr{O}})} \end{array}$$

and a constant $K_{\hat{\mathscr{O}}}$ such that

$$\left\|\left(I - (\mathscr{H}_n\mathscr{S}_n)^{-L}\right)\big|_{\mathscr{R}(\mathscr{H}_n\mathscr{S}_n\hat{\mathscr{O}})}\right\| \le \frac{K_{\hat{\mathscr{O}}}}{n}.$$

The proofs of this lemma and the next one are quite technical and involved, and thus are relegated to the appendix.

The next lemma takes care of approximating the direct feed-through operator $\hat{D}_{11}$, which is approximated by the direct feed-through operator $\overline{D}_n$ of $\overline{G}_n$.

Lemma 5: There is a constant $K_{\hat{D}}$ such that

$$\|\hat{D}_{11} - \overline{D}_n\| \leq \frac{K_{\hat{D}}}{n}.$$

Combining Proposition 3 and Lemma 5, we get that $\overline{T}_n$ approximates T by

$$\|T - \overline{T}_n\| \leq \frac{K_o}{n}\|\overline{T}_{on}\| + \frac{K_{\hat{D}}}{n}. \tag{13}$$

To get a bound with $\|\overline{T}_n\|$ on the right, note that $\overline{T}_n = \overline{T}_{on} + \overline{D}_n$, which implies by the triangle inequality that $\|\overline{T}_{on}\| - \|\overline{D}_n\| \leq \|\overline{T}_n\|$, and

$$\|\overline{T}_{on}\| \leq \|\overline{D}_n\| + \|\overline{T}_n\| \leq \|\hat{D}_{11}\| + \|\overline{T}_n\|.$$

Since $\|\hat{D}_{11}\|$ is a constant, combining with (13) yields

$$\|T - \overline{T}_n\| \leq \frac{K_o}{n}\|\hat{D}_{11}\| + \frac{K_o}{n}\|\overline{T}_n\| + \frac{K_{\hat{D}}}{n}.$$

Finally, since $\|T\| - \|\overline{T}_n\| \leq \|T - \overline{T}_n\|$, we get

$$\|T\| \leq \frac{K_o\|\hat{D}_{11}\| + K_{\hat{D}}}{n} + \left(1 + \frac{K_o}{n}\right)\|\overline{T}_n\| =: \frac{K_1}{n} + \left(1 + \frac{K_o}{n}\right)\|\overline{T}_n\|$$

and thus we have arrived at the main inequality (5).

VI. Geometrical Interpretations

In the previous section we gave an approximation procedure to obtain approximately optimal controllers. The procedure is based on forming an "approximate" finite-dimensional system to an infinite-dimensional one. A question may be asked as to whether the infinite-dimensional problem may be *exactly* reducible to a finite-dimensional l^1 problem. For example, in [1], the $\mathscr{H}^\infty$ sampled-data problem was treated by the lifting technique, and an exact reduction of the resulting infinite-dimensional problem to a finite-dimensional one is possible. This motivates the question as to whether a similar exact reduction is possible in the l^1 problem.

In this section, we will not give a definite answer to this question, but it is our purpose to illustrate some of the underlying geometry in the reduction, and to suggest that the l^1 sampled-data problem may not be exactly reducible to a finite-dimensional l^1 problem. We will give a geometric reasoning which shows that the fundamental difference between the reduction of the $\mathscr{H}^\infty$ and the l^1 sampled-data problems has to do with the difference between the geometry of finite-dimensional Hilbert and Banach spaces.

Let us go back to the formulation of the problem involving the infinite-dimensional generalized plant $\tilde{G}$, and consider the decomposition of $\tilde{G}$ in feedback with the controller C (Fig. 6).

To facilitate the geometric arguments we are about to make, we assume that the operator $\hat{D}_{11} = 0$. Note that this assumption is valid only when $G_{11} = 0$, and this is an unrealistic assumption for most interesting control problems, but the assumption is made for the purpose of illustration. With the assumption $\hat{D}_{11} = 0$, the decomposed system in feedback with C is shown in Fig. 8, where $\hat{\mathscr{C}} := [\hat{C}_1 \ \ \hat{D}_{12}]$.

We first look at possible decompositions of the output space $L^\infty[0,\tau]$. From Fig. 8, it is clear that

$$\mathscr{F}(\tilde{G}, C) = \hat{\mathscr{C}}\mathscr{F}(\tilde{G}_{oo}, C)\hat{B}_1$$

which means that the output signal $\hat{z}$ takes values in $\mathscr{R}(\hat{\mathscr{C}}) \subset L^\infty[0,\tau]$ (at each point in time). Since $\hat{\mathscr{C}}$: $\mathbb{R}^{n_x+n_u} \to L^\infty[0,\tau]$, then $\mathscr{R}(\hat{\mathscr{C}})$ is a finite-dimensional subspace of $L^\infty[0,\tau]$, and there exists a projection on it $\Pi_{\mathscr{R}(\hat{\mathscr{C}})}$: $L^\infty[0,\tau] \to \mathscr{R}(\hat{\mathscr{C}})$ [20]. By the definition of a projection, we have that for any $x \in \mathbb{R}^{n_x+n_u}$, $\|\hat{\mathscr{C}}x\|_{L^\infty[0,\tau]} = \|\Pi_{\mathscr{R}(\hat{\mathscr{C}})}\hat{\mathscr{C}}x\|_{\mathscr{R}(\hat{\mathscr{C}})}$, therefore

$$\left\|\Pi_{\mathscr{R}(\hat{\mathscr{C}})}\hat{\mathscr{C}}\mathscr{F}\left(\tilde{G}_{oo}, C\right)\hat{B}_1\right\| = \left\|\hat{\mathscr{C}}\mathscr{F}\left(\tilde{G}_{oo}, C\right)\hat{B}_1\right\| = \left\|\mathscr{F}(\tilde{G}, C)\right\|.$$

Note that $\Pi_{\mathscr{R}(\hat{\mathscr{C}})}\hat{\mathscr{C}}\mathscr{F}(\tilde{G}_{oo}, C)\hat{B}_1$ is a system with a finite-dimensional output space, namely $\mathscr{R}(\hat{\mathscr{C}})$, and the norm on $\mathscr{R}(\hat{\mathscr{C}})$ is the norm it inherits as a subspace of $L^\infty[0,\tau]$.

A similar reduction is possible with the input space, for this, we need to look at the preadjoint operators. Since for any Banach space operator A, $\|A\| = \|A^*\|$, we have that

$$\left\|\Pi_{\mathscr{R}(\hat{\mathscr{C}})}\hat{\mathscr{C}}\mathscr{F}\left(\tilde{G}_{oo}, C\right)\hat{B}_1\right\| = \left\|{}^*\hat{B}_1\,{}^*\mathscr{F}\left(\tilde{G}_{oo}, C\right){}^*\hat{\mathscr{C}}\,{}^*\Pi_{\mathscr{R}(\hat{\mathscr{C}})}\right\|$$

and as before, we can project on $\mathscr{R}({}^*\hat{B}_1) \subset \mathscr{L}^1[0,\tau]$ without changing the induced norm

$$\begin{aligned}\left\|{}^*\hat{B}_1{}^*\mathscr{F}\left(\tilde{G}_{oo}, C\right){}^*\hat{\mathscr{C}}{}^*\Pi_{\mathscr{R}(\hat{\mathscr{C}})}\right\| &= \left\|\Pi_{\mathscr{R}({}^*\hat{B}_1)}{}^*\hat{B}_1{}^*\mathscr{F}\left(\tilde{G}_{oo}, C\right){}^*\hat{\mathscr{C}}{}^*\Pi_{\mathscr{R}(\hat{\mathscr{C}})}\right\| \\ &= \left\|\Pi_{\mathscr{R}(\hat{\mathscr{C}})}\hat{\mathscr{C}}\mathscr{F}\left(\tilde{G}_{oo}, C\right)\hat{B}_1\Pi^*_{\mathscr{R}({}^*\hat{B}_1)}\right\|\end{aligned}$$

where the last equality follows by taking the adjoints. Also, note that since $\Pi_{\mathscr{R}({}^*\hat{B}_1)}$: $L^1[0,\tau] \to \mathscr{R}({}^*\hat{B}_1)$ then $\Pi^*_{\mathscr{R}({}^*\hat{B}_1)}$: $(\mathscr{R}({}^*\hat{B}_1))^* \to L^\infty[0,\tau]$, where $(\mathscr{R}({}^*\hat{B}_1))^*$ is the dual space of $\mathscr{R}({}^*\hat{B}_1)$, and it is finite-dimensional since $\mathscr{R}({}^*\hat{B}_1)$ is.

Combining the reduction on both the input and the output spaces, we have

$$\begin{aligned}\left\|\hat{\mathscr{C}}\mathscr{F}\left(\tilde{G}_{oo}, C\right)\hat{B}_1\right\| &= \left\|\Pi_{\mathscr{R}(\hat{\mathscr{C}})}\hat{\mathscr{C}}\mathscr{F}\left(\tilde{G}_{oo}, C\right)\hat{B}_1\Pi^*_{\mathscr{R}({}^*\hat{B}_1)}\right\| \\ &=: \left\|\mathscr{F}(\overline{G}, C)\right\|,\end{aligned} \tag{14}$$

where $\overline{G}$ is defined by

$$\overline{G} := \begin{bmatrix}\Pi_{\mathscr{R}(\hat{\mathscr{C}})} & 0 \\ 0 & I\end{bmatrix}\tilde{G}\begin{bmatrix}\Pi^*_{\mathscr{R}({}^*\hat{B}_1)} & 0 \\ 0 & I\end{bmatrix}.$$

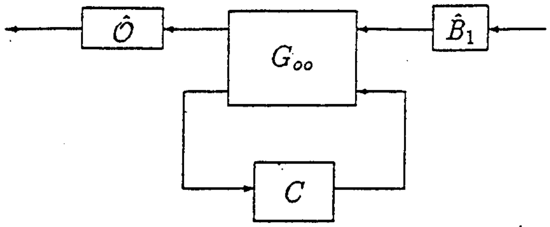

Fig. 8. Decomposition of $\tilde{G}$ with $\hat{D}_{11} = 0$.

Equation (14) shows that the original problem is reducible to the standard problem with the generalized plant $\overline{G}$. Since $\overline{G}$ has finite-dimensional input and output spaces (since $\mathscr{R}(\hat{\mathscr{O}})$ and $(\mathscr{R}(^*\hat{B}_1))^*$ are finite dimensional), we have arrived at an equivalent finite-dimensional problem. This problem is not necessarily a standard finite-dimensional l^1 problem, it is only so if the input and output spaces ($\mathscr{R}(\hat{\mathscr{O}})$ and $(\mathscr{R}(^*\hat{B}_1))^*$) are linearly isometrically isomorphic to an $l^\infty(n)$ space for some n.

Remark: In the $\mathscr{H}^\infty$ sampled-data problem, the situation is much simpler. In that case, $\mathscr{R}(\hat{\mathscr{O}})$ and $(\mathscr{R}(^*\hat{B}_1))^*$ as subspaces of $L^2[0,\tau]$, are immediately linearly isometric to Euclidean spaces (that is $l^2(n)$), since every finite-dimensional Hilbert space is linearly isometric to a Euclidean space of equal dimension.

Thus, the question arises as to what the spaces $\mathscr{R}(\hat{\mathscr{O}})$ and $(\mathscr{R}(^*\hat{B}_1))^*$ look like, and to whether they are isometric to $l^\infty(n)$? If the answer is affirmative, we can use this identification with $l^\infty(n)$ and obtain a generalized plant which has an $l^\infty(n)$ for each of its input and output spaces, and the problem then becomes a standard l^1 problem. However, the answer is negative. This can be seen by a simple example, where we plot the unit ball of the space $\mathscr{R}(\hat{\mathscr{O}})$ and show that there is no linear transformation that can transform it to a unit ball of an $l^\infty(n)$ space.

The example we consider is as follows: first recall that the operator $\hat{\mathscr{O}}$ is given by the following kernel function

$$\hat{\mathscr{O}}(t) := \begin{bmatrix} \hat{C}_1(t) & \tilde{D}_{12}(t) \end{bmatrix} = \begin{bmatrix} C_1 e^{At} & C_1\left(\int_0^t e^{As}\,ds\right) B_2 \end{bmatrix}.$$

We will consider the subspace $\mathscr{R}(\hat{C}_1) \subset \mathscr{R}(\hat{\mathscr{O}})$ and show that it cannot be a subspace of any $l^\infty(n)$. Recall that the norm on the space $\mathscr{R}(\hat{C}_1)$ is the norm inherited as a subspace of $L^\infty[0,\tau]$. The unit ball in $\mathscr{R}(\hat{C}_1)$ can be plotted by choosing a basis, and then computing the $L^\infty[0,\tau]$ norm for combinations of the basis elements. The particular example we pick is

$$A = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}; \qquad C = [1 \quad 1/2],$$

with $\tau = 1$. For this example $\mathscr{R}(\hat{C}_1)$ has dimension two, and a basis for it is given by

$$x_1(t) := C_1 e^{At}\begin{bmatrix} 1 \\ 0 \end{bmatrix}; \qquad x_2(t) := C_1 e^{At}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To plot the unit ball in $\mathscr{R}(\hat{C}_1)$, we represent any $x \in \mathscr{R}(\hat{C}_1)$ by $x = \alpha_1 x_1 + \alpha_2 x_2$. The ball in Fig. 9 represents

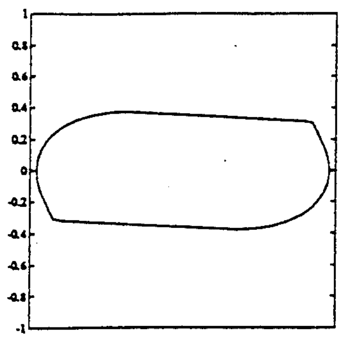

Fig. 9. The unit ball of $\mathscr{R}(\hat{C}_1)$.

$\|x\| = 1$, and the axes are α_1 and α_2. The unit ball in an $l^\infty(n)$ space is an n-cube, and the unit ball of any 2-dimensional subspace of $l^\infty(n)$ is a 2-dimensional "slice" through an n-cube, and it is clear that the boundary of this 2-dimensional cube must be made up of straight lines, i.e., it must be a polygon. Now, for $\mathscr{R}(\hat{C}_1)$ to be linearly isometric to a subspace of $l^\infty(n)$, a necessary condition is that its unit ball [that of $\mathscr{R}(\hat{C}_1)$] must be linearly transformable to a polygon, which means that it should itself be a polygon. Since the unit ball of the particular example in Fig. 9 is not a polygon, we conclude that $\mathscr{R}(\hat{C}_1)$ [and consequently $\mathscr{R}(\hat{\mathscr{O}})$] is not linearly isometrically isomorphic to an $l^\infty(n)$ space for any n.

We end this section with a geometric interpretation of the approximation procedure given previously. If we apply the approximation procedure to the system in Fig. 8, the result is the system

$$\mathscr{S}_n \hat{\mathscr{O}} \mathscr{F}\left(\tilde{G}_{oo}, C\right) \hat{B}_1 \mathscr{H}_n. \tag{15}$$

Looking only at the output side (the input side can be interpreted similarly using adjoints), the norm on the output side is essentially measured by sampling the elements in $\mathscr{R}(\hat{\mathscr{O}})$, that is, the norm of a function $f \in \mathscr{R}(\hat{\mathscr{O}})$ is computed by taking the $l^\infty(n)$ norm of n samples. As before, we can plot the unit ball of $\mathscr{R}(\hat{C}_1)$ in this new norm which we will call the "samples norm." (Actually, we will plot the coefficients α_1, α_2, hence the plot is two dimensional). This norm approximates the actual norm on $\mathscr{R}(\hat{C}_1)$ for large n. This approximation can be seen in Fig. 10 (for $n = 3$), where the samples norm unit ball is superimposed over the actual unit ball of $\mathscr{R}(\hat{C}_1)$. It is interesting to see that what is being done, is approximation of the unit ball of $\mathscr{R}(\hat{\mathscr{O}})$ by polygons. Thus the approximation procedure for solving the sampled-data problem can be interpreted as an approximation of norms of the input and output spaces. It is interesting to note here that the unit balls of $\mathscr{R}(\hat{\mathscr{O}})$ and $(\mathscr{R}(^*\hat{B}_1))^*$, generally represent nonlinear constraints, very much as in the continuous-time L^1 problem [6], while in discrete-time l^1 problems, the constraints are always linear. Therefore, the fact that the norms in the sampled-data problem represent nonlinear constraints (roughly speaking), seems to be a consequence of the continuous-time nature of the problem (just as in the L^1 problem). However, by essentially

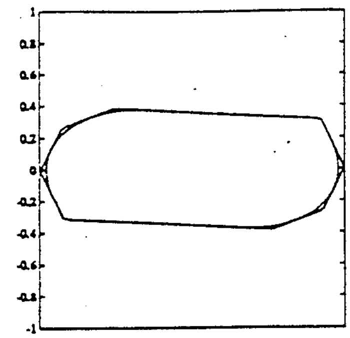

Fig. 10. The unit balls of $\mathscr{R}(\hat{C}_1)$ with the actual, and the samples norms.

approximating the nonlinear constraints by linear ones, we are able to reduce the problem to a standard discrete-time l^1 problem.

Finally, we point out that the mathematical reason behind the difference in the reductions of the $\mathscr{H}^\infty$ and l^1 sampled-data problems, is that in the former, any finite-dimensional Hilbert space is linearly isometric to $l^2(n)$, while in the latter, not every finite-dimensional Banach space is linearly isometric to $l^\infty(n)$. This reflects the fact that the isometric class of Banach spaces of dimension n is a much richer class (there is an infinite number of them, for example $l^p(n)$ for $1 \le p \le \infty$), than the class of Hilbert spaces of dimension n [of which there is only one, $l^2(n)$].

VII. Conclusions

This paper provides a solution for the sampled-data l^1 problem through approximation. Utilizing lifting techniques, the input/output map is decomposed in such a way that the infinite-dimensional part of the system is isolated independently of the controller. This part is then approximated in a precise way by a finite-dimensional system, whose dimension can be determined given any degree of accuracy. Computable bounds on the norm of the difference of the actual system and the approximated system are furnished, and they all depend entirely on the system's data. It is shown that the rate of convergence of this approximation is $(1/n)$.

It is interesting to note that the same approach and approximation arguments in this paper can be followed to obtain bounds like the main inequality for the L^1-induced norm sampled-data problem. A combination of this with the Riesz–Thorin convexity theorem would then show that the main inequality (with different constants) holds for general L^p-induced norm problems. In particular this holds for the L^2-induced norm case. In this case, this approximation procedure was shown to converge in [15]. The results of this paper and the above convexity argument indicate that stronger convergence at the $(1/n)$ rate actually holds. However, for the case of the L^2-induced norm sampled-data problem, an exact equivalence to a discrete-time problem can be obtained [1]. It is indicated in this paper by geometric arguments that this exact correspondence may not be possible in general for L^∞-induced norm sampled-data problems.

The approach followed in this paper is readily applicable to the structured perturbations problem for sampled-data systems [16]. The minimization problem in this set-up involves spectral radius functions, and a similar result follows from the continuity of the spectral radius function. The derivation of explicit bounds takes more work and will be reported elsewhere.

Appendix A

In the following proofs it is assumed for simplicity that the matrices D_{11} and D_{12} are zero. If D_{12} is not zero, the statement of Lemma 4 still holds. If D_{11} is not zero, the statement of Lemma 5 does not hold, however the main inequality does hold but has to be derived differently.

Proof of Lemma 4

a) If $f \in \mathscr{R}_{(^*\hat{B}_1)}$, then $f(t) = {}^*\hat{B}_1(t)x = B_1' e^{A'(\tau - t)}x$, for some $x \in \mathbb{R}^{n_x}$. We may assume without loss of generality that (A, B_1) is controllable, since if not, we can decompose the state space into the controllable and uncontrollable subspaces, and write

$$ {}^*\hat{B}_1(t) = [B_c' \quad 0] e^{\begin{bmatrix} A_c' & 0 \\ ? & A_{nc}' \end{bmatrix}(\tau - t)} T, $$

where (A_c, B_c) is controllable, T is nonsingular, and then note that $\mathscr{R}_{(^*\hat{B}_1)}$ is the same as the range of $\{B_c' e^{A_c'(\tau - t)}\}$, and thus work with (A_c, B_c) instead of (A, B_1). We also note that since the eigenvalues of A_c are a subset of the eigenvalues of A, then if τ/n is nonpathological for A, it is nonpathological for A_c.

Now, to show that $(\mathscr{H}_n \mathscr{S}_n)$ has a left inverse, we need to show that $(\mathscr{H}_n \mathscr{S}_n)$: $\mathscr{R}_{(^*\hat{B}_1)} \to L^1[0, \tau]$ is injective, but since $\mathscr{H}_n$: $l^1(n) \to L^1[0, \tau]$ is injective, it suffices to show that $\mathscr{S}_n$: $\mathscr{R}_{(^*\hat{B}_1)} \to l^1(n)$ is injective, or equivalently, that it has no null space. Given $f \in \mathscr{R}_{(^*\hat{B}_1)}$, let $\bar{f} := \mathscr{S}_n f$, since $f(t) = B_1' e^{A'(\tau - t)}x$ for some $x \in \mathbb{R}^{n_x}$, then

$$ \begin{aligned} \bar{f}_i &= \frac{n}{\tau} \int_{i\tau/n}^{(i+1)\tau/n} B_1' e^{A'(\tau - t)} x \, dt \\ &= \frac{n}{\tau} B_1' \int_0^{\tau/n} e^{A'(\tau/n - \hat{t})} \, d\hat{t} \, e^{A'(n-i-1)\tau/n} x \\ &= \frac{n}{\tau} B_1' \Psi'(\tau/n) e^{A'(n-i-1)\tau/n} x, \end{aligned} $$

or in matrix notation

$$ \begin{bmatrix} \bar{f}_0 \\ \vdots \\ \bar{f}_{n-1} \end{bmatrix} = \frac{n}{\tau} \begin{bmatrix} B_1' \Psi'(\tau/n) e^{A'(n-1)\tau/n} \\ \vdots \\ B_1' \Psi'(\tau/n) \end{bmatrix} x =: \frac{n}{\tau} \mathscr{B}_n' x. \quad (16) $$

Note that for $n \ge n_x$, $\mathscr{B}_n$ contains the controllability matrix of $(e^{A\tau/n}, \Psi(\tau/n)B_1)$, and since (A, B_1) is controllable and τ/n is a nonpathological sampling period, then $(e^{A\tau/n}, \Psi(\tau/n)B_1)$ is controllable, and thus the matrix $\mathscr{B}_n'$

has full rank. Therefore, if $f \in \mathscr{R}_{(*\hat{B}_1)}$, $f \neq 0$, then $f = {}^*\hat{B}_1 x$, for some $x \in \mathbb{R}^{n_x}$, $x \neq 0$, consequently $\tilde{f} \neq 0$ (since $\mathscr{B}_n'$ has full rank), implying that $\mathscr{T}_n$ has no null space and thus is injective.

To obtain the bounds we need, it is necessary to bound the norm of x that solves the equation $\tilde{f} = \mathscr{B}_n' x$ by the norm of $\tilde{f}$. Since $\mathscr{B}_n'$ has full rank (as a matrix), there exists a constant c_1 such that if $\tilde{f} = (n/\tau)\mathscr{B}_n' x$ then $(n/\tau)\|x\|_1 \le c_1 \|\tilde{f}\|_{l^1(n)}$ (where $\|x\|_1$ is the 1-norm on $\mathbb{R}^{n_x}$). The constant c_1 can be taken as the norm of the left inverse to $\mathscr{B}_1'$. See the appendix for the proof that c_1 is independent of n.

If we define $\bar{f} := \mathscr{H}_n \tilde{f}$, then we have from the definition of $\mathscr{H}_n$: $l^1(n) \to L^1[0,\tau]$ that $\|\bar{f}\|_{L^1[0,\tau]} = (\tau/n)\|\tilde{f}\|_{l^1(n)}$. Combining this with the previous bound yields that for $\bar{f} = (\mathscr{H}_n\mathscr{T}_n)^* \hat{B}_1 x$

$$\|x\|_1 \le c_1 \|\bar{f}\|_{L^1[0,\tau]}.$$

Now, to compute a bound on $\|(I - (\mathscr{H}_n\mathscr{T}_n)^{-L})|_{\mathscr{R}(\mathscr{H}_n\mathscr{T}_n{}^*\hat{B}_1)}\|$, let $\bar{f}$ be an element in $\mathscr{R}_{(\mathscr{H}_n\mathscr{T}_n{}^*\hat{B}_1)}$, i.e., $\bar{f} = \mathscr{H}_n\mathscr{T}_n^* \hat{B}_1 x$ for some $x \in \mathbb{R}^{n_x}$. We have already shown the existence of the left inverse $(\mathscr{H}_n\mathscr{T}_n)^{-L}$, by its definition $(\mathscr{H}_n\mathscr{T}_n)^{-L}\bar{f} = {}^*\hat{B}_1 x$, therefore

$$\begin{aligned}
&\|(I - (\mathscr{H}_n\mathscr{T}_n)^{-L})\bar{f}\|_{L^1[0,\tau]} \\
&= \|\mathscr{H}_n\mathscr{T}_n^*\hat{B}_1 x - {}^*\hat{B}_1 x\|_{L^1[0,\tau]} \\
&= \int_0^\tau \|(\mathscr{H}_n\mathscr{T}_n^*\hat{B}_1 x)(t) - ({}^*\hat{B}_1 x)(t)\|\, dt \\
&= \sum_{i=1}^{n-1} \int_{i\tau/n}^{(i+1)\tau/n} \|(\mathscr{H}_n\mathscr{T}_n^*\hat{B}_1 x)(t) - ({}^*\hat{B}_1 x)(t)\|\, dt \\
&= \sum_{i=1}^{n-1} \int_{i\tau/n}^{(i+1)\tau/n} \Big\| \frac{n}{\tau}\Big(\int_{i\tau/n}^{(i+1)\tau/n} {}^*\hat{B}_1(s)\, ds\Big) x \\
&\qquad - {}^*\hat{B}_1(t) x\Big\|\, dt \\
&\le \sum_{i=1}^{n-1} \int_{i\tau/n}^{(i+1)\tau/n} \Big\| \frac{n}{\tau}\Big(\int_{i\tau/n}^{(i+1)\tau/n} {}^*\hat{B}_1(s)\, ds\Big) \\
&\qquad - {}^*\hat{B}_1(t)\Big\|\, dt \|x\|_1 \\
&\le \sum_{i=1}^{n-1} \frac{\tau^2}{n^2} \sup_{0\le t\le\tau} \Big\|\frac{d^*\hat{B}_1(t)}{dt}\Big\| \|x\|_1 \\
&\le \frac{\tau^2}{2n^2}\|x\|_1 \sum_{i=1}^{n-1} \sup_{0\le t\le\tau} \|B_1' A' e^{A'(\tau - t)}\| \\
&\le \frac{\tau^2}{2n^2} c_1 \|\bar{f}\| n \|B_1'\|\, \|A'\| e^{\|A'\|\tau} \\
&\le \frac{\tau^2}{2} c_1 \|B_1'\|\, \|A'\| e^{\|A'\|\tau} \frac{1}{n} \|\bar{f}\|
\end{aligned} \tag{17}$$

see (18) which means that

$$\begin{aligned}
&\|(I - (\mathscr{H}_n\mathscr{T}_n)^{-L})\|_{\mathscr{R}(\mathscr{H}_n\mathscr{T}_n{}^*\hat{B}_1)}| \\
&\qquad \le \frac{\tau^2}{2} c_1 \|B_1'\|\, \|A'\| e^{\|A'\|\tau} \frac{1}{n} =: \frac{K_{\hat{B}}}{n}.
\end{aligned}$$

Proof of b): By definition, $\hat{\mathscr{C}} := [\hat{C}_1 \;\; \tilde{D}_{12}]$, and

$$\begin{aligned}
\hat{\mathscr{C}}(t) &= [\hat{C}_1(t) \;\; \tilde{D}_{12}(t)] = \Big[C_1 e^{At} \;\; C_1\Big(\int_0^t e^{As}\, ds\Big) B_2\Big] \\
&= [0 \;\; C_1] e^{\begin{bmatrix} 0 & 0 \\ I & A \end{bmatrix} t} \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix},
\end{aligned}$$

where the last equality is a consequence of the formula $\int_0^t e^{As} ds = [0 \;\; I] e^{\begin{bmatrix} 0 & 0 \\ I & A \end{bmatrix} t} \begin{bmatrix} I \\ 0 \end{bmatrix}$. With an argument similar to that in the proof of part a), we can replace $[0 \;\; C_1]$ and $\begin{bmatrix} 0 & 0 \\ I & A \end{bmatrix}$ by C_o and A_o such that (C_o, A_o) is observable, i.e.,

$$\begin{aligned}
[0 \;\; C_1] e^{\begin{bmatrix} 0 & 0 \\ I & A \end{bmatrix} t} \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} &= [C_0 \;\; 0] e^{\begin{bmatrix} A_o & 0 \\ ? & A_{no} \end{bmatrix} t} T \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} \\
&= [C_o e^{A_o t} \;\; 0] \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = C_0 e^{A_o t} R_1,
\end{aligned}$$

where $\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} := T \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}$. Furthermore, we can replace R_1 by B_f, which is made up of the linearly independent columns of R_1, and define $\hat{\mathscr{C}}_o(t) := C_o e^{A_o t} B_f$, we then have

$$\begin{aligned}
\mathscr{R}(\hat{\mathscr{C}}) &= \mathscr{R}\Big(\Big\{[0 \;\; C_1] e^{\begin{bmatrix} 0 & 0 \\ I & A \end{bmatrix} t} \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}\Big\}\Big) \\
&= \mathscr{R}(\{C_o e^{A_o t} B_f\}) = \mathscr{R}(\hat{\mathscr{C}}_o).
\end{aligned}$$

Now, to show the existence of $(\mathscr{H}_n\mathscr{S}_n)^{-L}$ on $\mathscr{R}_{(\mathscr{H}_n\mathscr{S}_n\hat{\mathscr{C}})}$, or equivalently, that $(\mathscr{H}_n\mathscr{S}_n)$ is injective, it suffices to show that $\mathscr{S}_n$ has no null space in $\mathscr{R}_{(\hat{\mathscr{C}})}$ (since $\mathscr{H}_n$: $l^\infty(n) \to L^\infty[0,\tau]$ is injective). By the representation above, if $f \in \mathscr{R}_{(\hat{\mathscr{C}})}$, $f \neq 0$, then $f(t) = C_o e^{A_o t} B_f x$ for some $x \neq 0$, $x \in \mathbb{R}^p$ (where $p \le n_x + n_u$). Let $\tilde{f} := \mathscr{S}_n f$, then $\tilde{f}_i = C_o e^{A_o i\tau/n} B_f x$, or in matrix notation

$$\begin{bmatrix} \tilde{f}_0 \\ \vdots \\ \tilde{f}_{n-1} \end{bmatrix} = \begin{bmatrix} C_o \\ \vdots \\ C_o (e^{A_o \tau/n})^{n-1} \end{bmatrix} B_f x =: \mathscr{C}_n B_f x.$$

Since (C_o, A_o) is observable and τ/n is not pathological, then $(C_o, e^{A_o\tau/n})$ is observable implying that the matrix $\mathscr{C}_n$ has full column rank (for $n \ge 2n_x$), and since B_f also has column rank, then $\tilde{f} \neq 0$, which shows that $\mathscr{S}_n$ has no null space in $\mathscr{R}_{(\hat{\mathscr{C}})}$.

To obtain the bounds we need, it is necessary to have a bound on the norm $\|x\|_\infty$ ($\|\cdot\|_\infty$ is the maximum component norm in $\mathbb{R}^p$) of solutions of the equation $\tilde{f} = \mathscr{C}_n B_f x$. Since both $\mathscr{C}_n$ and B_f have full column rank, they both have left inverses $\mathscr{C}_n^{-L}$, B_f^{-L}, and

$$\|x\|_\infty \le \|B_f^{-L}\|\, \|\mathscr{C}_n^{-L}\|\, \|\tilde{f}\|_{l^\infty(n)}.$$

Since $\mathscr{H}_n: l^\infty(n) \to L^\infty[0,\tau]$ preserves norms, that is, for $\bar{f} := \mathscr{H}_n \tilde{f} = (\mathscr{H}_n \mathscr{S}_n) f$, we have that $\|\bar{f}\|_{L^\infty[0,\tau]} = \|f\|_{l^\infty(n)}$, the above bound becomes

$$\|x\|_\infty \le c_2 \|\bar{f}\|_{L^\infty[0,\tau]}.$$

The proof that the bound c_2 is independent of n, though long, is entirely similar to that for c_1 in part a).

Now let $f \in \mathscr{R}_{(\hat{\mathscr{O}})}$, therefore $f \in \mathscr{R}_{(\hat{\mathscr{O}}_o)}$ which means that $f = \hat{\mathscr{O}}_o x$ for some $x \in \mathbb{R}^p$. Let $\bar{f} := (\mathscr{H}_n \mathscr{S}_n) f$, by the definition of $(\mathscr{H}_n \mathscr{S}_n)^{-L}$, we have that $(\mathscr{H}_n \mathscr{S}_n)^{-L} f = f = \hat{\mathscr{O}}_o x$. We now compute,

$$\begin{aligned}
&\left\|\left(I - (\mathscr{H}_n\mathscr{S}_n)^{-L}\right)\bar{f}\right\|_{L^\infty[0,\tau]} \\
&= \sup_{0\le t\le\tau} \left\|\left(\mathscr{H}_n\mathscr{S}_n\hat{\mathscr{O}}_o x\right)(t) - \left(\hat{\mathscr{O}}_o x\right)(t)\right\| \\
&= \sup_{0\le i\le n-1}\ \sup_{0\le \hat{t}\le \tau/n} \left\|\left(\mathscr{H}_n\mathscr{S}_n\hat{\mathscr{O}}_o x\right)(\hat{t} + i\tau/n) - \left(\hat{\mathscr{O}}_o x\right)(\hat{t} + i\tau/n)\right\| \\
&= \sup_{0\le i\le n-1}\ \sup_{0\le \hat{t}\le \tau/n} \left\|\left(\hat{\mathscr{O}}_o x\right)(i\tau/n) - \left(\hat{\mathscr{O}}_o x\right)(\hat{t} + i\tau/n)\right\| \\
&\le \sup_{0\le i\le n-1}\ \sup_{0\le \hat{t}\le \tau/n} \left\|\int_{i\tau/n}^{\hat{t}+i\tau/n} \frac{d\hat{\mathscr{O}}_o}{ds}(s)\,ds\right\| \|x\|_\infty \\
&\le \sup_{0\le i\le n-1}\ \sup_{0\le \hat{t}\le \tau/n} \int_{i\tau/n}^{\hat{t}+i\tau/n} \left\|\frac{d\hat{\mathscr{O}}_o}{ds}(s)\right\| ds \|x\|_\infty \\
&\le \sup_{0\le i\le n-1} \int_{i\tau/n}^{(i+1)\tau/n} \left\|\frac{d\hat{\mathscr{O}}_o}{ds}(s)\right\| ds \|x\|_\infty \\
&\le \sup_{0\le i\le n-1}\ \sup_{0\le s\le \tau} \left\|\frac{d\hat{\mathscr{O}}_o}{ds}(s)\right\| \frac{\tau}{n}\|x\|_\infty \\
&\le \sup_{0\le i\le n-1}\ \sup_{0\le s\le \tau} \|C_o\|\,\|A_o\| e^{\|A_o\|s} \|B_f\| \frac{\tau}{n}\|x\|_\infty \\
&\le \|C_o\|\,\|A_o\| e^{\|A_o\|\tau} \|B_f\| \frac{\tau}{n} c_2 \|\bar{f}\|_{L^\infty[0,\tau]},
\end{aligned}$$

which results in

$$\left\|\left(I - (\mathscr{H}_n\mathscr{S}_n)^{-L}\right)\big|_{\mathscr{R}(\mathscr{H}_n\mathscr{S}_n\hat{\mathscr{O}})}\right\| \le \|C_o\|\,\|A_o\| e^{\|A_o\|\tau}\|B_f\| c_2 \tau \frac{1}{n} =: \frac{K_{\hat{\mathscr{O}}}}{n}. \quad \blacksquare$$

Proof of Lemma 5

If $\hat{D}_{11}$ comes from the lifting of a MIMO G_{11}, then $\hat{D}_{11}$ operates on vector signals, i.e., $\hat{D}_{11}: L_n^\infty[0,\tau] \to L_m^\infty[0,\tau]$. The induced norm of such an operator is bounded above by the maximum row sum of the matrix of the $L^\infty[0,\tau]$-induced norms of the SISO subsystems. We will prove the lemma as if $\hat{D}_{11}$ is scalar, the MIMO statement follows from the fact that if each entry in the matrix of norms tends to 0 separately, then the maximum row sum will also tend to zero.

The $L^\infty[0,\tau]$-induced norm of an operator $\mathscr{A}$ given by a kernel function $\mathscr{A}(t,s)$ is

$$\|\mathscr{A}\| = \sup_{0\le t\le \tau} \int_0^\tau |\mathscr{A}(t,s)|\,ds.$$

The kernel function of $\hat{D}_{11}$ is given from (1) by

$$\hat{D}_{11}(t,s) = C_1 e^{A(t-s)} \mathbf{1}_{(t-s)} B_1.$$

The operator $\overline{D}_n := (\mathscr{H}_n\mathscr{S}_n)\hat{D}_{11}(\mathscr{H}_n\mathscr{S}_n)$ has a kernel function which is piecewise constant over squares of width τ/n in $[0,\tau]\times[0,\tau]$, in particular, for $t = \hat{t} + i\tau/n$ and $s = \hat{s} + j\tau/n$, $\hat{t},\hat{s} \in [0,\tau/n]$

$$\overline{D}_n(t,s) = \frac{n}{\tau} C_1 e^{Ai\tau/n}\left(\int_{j\tau/n}^{(j+1)\tau/n} e^{-Ar}\,dr\right)\mathbf{1}_{(i-j-1)}B_1,$$

where $\mathbf{1}_{(\cdot)}$ is the unit step function with a discrete parameter. We now compute

$$\begin{aligned}
&\|\hat{D}_{11} - \overline{D}_n\| \\
&= \sup_{0\le t\le\tau} \int_0^\tau \left|\hat{D}_{11}(t,s) - \overline{D}_n(t,s)\right| ds \\
&= \sup_{0\le i\le n-1}\ \sup_{0\le t\le \tau/n} \sum_{j=0}^{n-1} \int_{j\tau/n}^{(j+1)\tau/n} \cdot\left|\hat{D}_{11}(t,s) - \overline{D}_n(t,s)\right| ds \\
&= \sup_i \sup_{\hat{t}} \sum_{j=0}^{n-1} \int_{j(\tau/n)}^{(j+1)\tau/n} \cdot\left|C_1\left(e^{A(i(\tau/n)+\hat{t}-s)}\mathbf{1}_{(t-s)} - \frac{n}{\tau}e^{Ai(\tau/n)} \cdot \int_{j(\tau/n)}^{(j+1)\tau/n} e^{-Ar}\,dr\,\mathbf{1}_{(i-j-1)}\right)B_1\right| ds \\
&\le \|C_1\|\,\|B_1\| \sup_i e^{\|A\|i(\tau/n)} \sup_{\hat{t}} \sum_{j=0}^{n-1} \int_{j(\tau/n)}^{(j+1)\tau/n} \cdot\left\|e^{A(\hat{t}-s)}\mathbf{1}_{(t-s)} - \frac{n}{\tau} \cdot \int_{j(\tau/n)}^{(j+1)\tau/n} e^{-Ar}\,dr\,\mathbf{1}_{(i-j-1)}\right\| ds \\
&\le \|C_1\|\,\|B_1\| e^{\|A\|\tau} \sup_i \sup_{\hat{t}} \cdot\left\{\sum_{j=0}^{i-1}\int_{j(\tau/n)}^{(j+1)\tau/n} \left\|e^{A(\hat{t}-s)} - \frac{n}{\tau} \cdot \int_{j(\tau/n)}^{(j+1)\tau/n} e^{-Ar}\,dr\right\| ds + \int_0^{\tau/n} \left\|e^{A(\hat{t}-s)}\right\| d\hat{s}\right\},
\end{aligned}$$

where the last term represents the case $i = j$. From (20) in Appendix B, we can bound

$$\int_{j(\tau/n)}^{(j+1)\tau/n} \| e^{A(\hat{t}-s)} - \frac{n}{\tau}\int_{j(\tau/n)}^{(j+1)\tau/n} e^{-Ar}\, dr \|\, ds$$

$$\leq \frac{\tau}{n}\| e^{A(\hat{t}-j(\tau/n))} - e^{-Aj(\tau/n)}\|$$

$$+ \frac{1}{2}\left[\sup_{j(\tau/n)\leq s\leq (j+1)\tau/n} \| Ae^{A(\hat{t}-s)}\| + \sup_{j(\tau/n)\leq r\leq (j+1)\tau/n} \| Ae^{-Ar}\| \right]\frac{\tau^2}{n^2}$$

$$\leq \frac{\tau}{n} e^{\|A\| j(\tau/n)}\| e^{A\hat{t}} - I\| + \|A\| e^{\|A\|\tau}\frac{\tau^2}{n^2}$$

$$\leq e^{\|A\|\tau}\left(\frac{\tau}{n}(e^{\|A\|\hat{t}} - 1) + \|A\|\frac{\tau^2}{n^2}\right).$$

Substituting back yields

$$\|\hat{D}_{11} - \overline{D}_n\| \leq \|C_1\|\,\|B_1\| e^{2\|A\|\tau} \sup_i \sup_{\hat{t}} \cdot\left\{ \sum_{j=0}^{i-1}\left(\frac{\tau}{n}(e^{\|A\|\hat{t}} - 1) + \|A\|\frac{\tau^2}{n^2}\right) + \frac{\tau}{n}\right\}$$

$$\leq \|C_1\|\,\|B_1\| e^{2\|A\|\tau} \cdot\left\{ \frac{\tau}{n}(e^{\|A\|\tau} - 1) + \|A\|\frac{\tau^2}{n} + \frac{\tau}{n}\right\}$$

$$=: \frac{K_{\hat{D}}}{n}$$ ■

Appendix B

Integral Inequalities

Let $F(t), F_1(t), F_2(t)$ be differentiable matrix valued functions. Some useful bounds shown below can be established by using the formula

$$F(t) = F(a) + \int_a^t \frac{dF}{dt}(s)\, ds,$$

and some manipulations involving cancelling common factors and bounding the norm of an integral by the integral of the norms. Note that in the following bounds, $\|\cdot\|$ is any matrix norm provided that the same norm is used on both sides of the same inequality,

$$\int_a^b \| \frac{1}{b-a}\left(\int_a^b F(s)\, ds\right) - F(t)\|\, dt \leq \frac{(b-a)^2}{2}\left(\sup_{a\leq t\leq b} \|\frac{dF}{dt}(t)\|\right) \tag{18}$$

$$\|\int_0^T F(t)F'(t)\, dt - \frac{1}{T}\left(\int_0^T F(s)\, ds\right)\left(\int_0^T F'(r)\, dr\right)\| \leq 2T^3\left(\sup_{0\leq t\leq T}\|\frac{dF}{dt}\|\right)^2 \tag{19}$$

$$\int_a^b \| F_1(t) - \frac{1}{b-a}\left(\int_a^b F_2(r)\, dr\right)\|\, dt \leq |b-a|\,|F_1(a) - F_2(a)| + \frac{1}{2}\left(\sup_{a\leq t\leq b}\|\frac{dF_1}{dt}(t)\| + \sup_{a\leq t\leq b}\|\frac{dF_2}{dt}(t)\|\right) \cdot |b-a|^2. \tag{20}$$

Completion of Proof of Lemma 4-a)

Claim: c_1 is independent of n.

Proof: We will construct c_1 as an upper bound on the norm of the left inverse to $\mathscr{B}_n'$. This is done by taking the pseudo-inverse as a left inverse to $\mathscr{B}_n'$, and finding a bound on its norm that is independent of n. The pseudo-inverse to $\mathscr{B}_n'$ is $(\mathscr{B}_n\mathscr{B}_n')^{-1}\mathscr{B}_n$, and note that the inverse exists since $\mathscr{B}_n'$ has full column rank. We first bound $\|(\mathscr{B}_n\mathscr{B}_n')^{-1}\|$. From the definition of $\mathscr{B}_n'$, we have

$$\mathscr{B}_n\mathscr{B}_n' = \sum_{i=0}^{n-1} e^{Ai\tau/n}\Psi(\tau/n)B_1B_1'\Psi'(\tau/n)e^{A'i\tau/n}.$$

Denote the controllability Grammian over the finite time τ, by

$$W_\tau := \int_0^\tau e^{At}B_1B_1'e^{A't}\, dt.$$

We will first show that $(n/\tau)(\mathscr{B}_n\mathscr{B}_n')\overset{n\to\infty}{\to} W_\tau$

$$\begin{aligned}
&\|W_\tau - \frac{n}{\tau}(\mathscr{B}_n\mathscr{B}_n')\|\\
&= \|\int_0^\tau e^{At}B_1B_1'e^{A't}\,dt\\
&\quad - \frac{n}{\tau}\sum_{i=0}^{n-1} e^{Ai\tau/n}\Psi(\tau/n)B_1B_1'\Psi'(\tau/n)e^{A'i\tau/n}\|\\
&= \|\sum_{i=0}^{n-1}\int_{i\tau/n}^{(i+1)\tau/n} e^{At}B_1B_1'e^{A't}\,dt\\
&\quad - \frac{n}{\tau}\sum_{i=0}^{n-1} e^{Ai\tau/n}\Psi(\tau/n)B_1B_1'\Psi'(\tau/n)e^{A'i\tau/n}\|\\
&\le \sum_{i=0}^{n-1}\|e^{Ai\tau/n}\left(\int_0^{\tau/n} e^{A\hat{t}}B_1B_1'e^{A'\hat{t}}\,d\hat{t}\right)e^{A'i\tau/n}\\
&\quad - \frac{n}{\tau}e^{Ai\tau/n}\Psi(\tau/n)B_1B_1'\Psi'(\tau/n)e^{A'i\tau/n}\|\\
&\le \sum_{i=0}^{n-1} e^{2\|A\|i\tau/n}\|\int_0^{\tau/n} e^{A\hat{t}}B_1B_1'e^{A'\hat{t}}\,d\hat{t}\\
&\quad - \frac{n}{\tau}\left(\int_0^{\tau/n} e^{As}\,ds\right)B_1B_1'\left(\int_0^{\tau/n} e^{A'r}\,dr\right)\|\\
&\le \sum_{i=0}^{n-1} e^{2\|A\|i\tau/n}2\frac{\tau^3}{n^3}\\
&\quad \cdot\left(\sup_{0\le\hat{t}\le\tau/n}\|B_1\|\,\|A\|e^{\|A\|\hat{t}}\right)^2 \qquad (21)
\end{aligned}$$

where the last step is a consequence of formula (19). After bounding $e^{2\|A\|i\tau/n} \le e^{2\|A\|\tau}$ and summing to yield a factor of n, (21) becomes

$$\|W_\tau - \frac{n}{\tau}(\mathscr{B}_n\mathscr{B}_n')\| \le 2\,e^{4\|A\|\tau}\|B_1\|^2\|A\|^2\frac{\tau^3}{n^2} =: M_1\frac{1}{n^2},$$

where M_1 is a constant. Now, since $(n/\tau)(\mathscr{B}_n\mathscr{B}_n')\overset{n\to\infty}{\to} W_\tau$, it follows that $((n/\tau)\mathscr{B}_n\mathscr{B}_n')^{-1}\overset{n\to\infty}{\to} W_\tau^{-1}$ [18, theorem 10.12]. An explicit bound (for large n) on the norm of $((n/\tau)\mathscr{B}_n\mathscr{B}_n')^{-1}$ in terms of the norm of W_τ^{-1} can be constructed in several ways, one way is by [18, theorem 10.11]

$$\begin{aligned}
\|\left(\frac{n}{\tau}\mathscr{B}_n\mathscr{B}_n'\right)^{-1}\| &\le \|W_\tau^{-1}\| + \|W_\tau^{-1}\|^2\|W_\tau - \left(\frac{n}{\tau}\mathscr{B}_n\mathscr{B}_n'\right)\|\\
&\quad + 2\|W_\tau^{-1}\|^3\|W_\tau - \left(\frac{n}{\tau}\mathscr{B}_n\mathscr{B}_n'\right)\|^2\\
&\le \|W_\tau^{-1}\| + \|W_\tau^{-1}\|^2M_1 + 2\|W_\tau^{-1}\|^3M_1^2\\
&=: M_2,
\end{aligned}$$

for $n^2 > 2M_1\|W_\tau^{-1}\|$. To take care of the case of n such that $n^2 \le 2M_1\|W_\tau^{-1}\|$, note that is only a finite number of such n's, and let M_3 be the maximum of $\|((n/\tau)\mathscr{B}_n\mathscr{B}_n')^{-1}\|$ over this finite set of n's (note also that $\|(\mathscr{B}_n\mathscr{B}_n')^{-1}\|$ exists if $n \ge n_x$ and τ/n is not a pathological sampling period). Letting $M_4 := \max\{M_2, M_3\}$, we obtain

$$\|\left(\frac{n}{\tau}\mathscr{B}_n\mathscr{B}_n'\right)^{-1}\| \le M_4 \Rightarrow \|(\mathscr{B}_n\mathscr{B}_n')^{-1}\| \le \frac{n}{\tau}M_4$$

$\forall n \ge n_x$ such that τ/n is not pathological. Finally, to find $\|(\mathscr{B}_n\mathscr{B}_n')^{-1}\mathscr{B}_n\|$, note that this is the induced norm from $l^1(n)$ to $\mathbb{R}^{n_x}$ with the $\|\cdot\|_1$ norm, i.e., it is the maximum column sum norm on the matrix, therefore

$$\begin{aligned}
\|(\mathscr{B}_n\mathscr{B}_n')^{-1}\mathscr{B}_n\| &\le \|(\mathscr{B}_n\mathscr{B}_n')^{-1}\|\\
&\quad\cdot\|\Psi(\tau/n)\left[(e^{A\tau/n})^{n-1}B_1\ \cdots\ B_1\right]\|\\
&\le \|(\mathscr{B}_n\mathscr{B}_n')^{-1}\|\|\Psi(\tau/n)\|\max\\
&\quad\cdot\left\{\|(e^{A\tau/n})^{n-1}\|,\cdots,\|e^{A\tau/n}\|\right\}\|B_1\|\\
&\le \frac{n}{\tau}M_4\frac{\tau}{n}e^{\|A\|\tau/n}\,e^{\|A\|\tau}\|B_1\|\\
&\le M_4\,e^{2\|A\|\tau}\|B_1\| =: c_1,
\end{aligned}$$

since $\|\Psi(\tau/n)\| = \|\int_0^{\tau/n} e^{As}\,ds\| \le \int_0^{\tau/n} e^{\|A\|s}\,ds \le \int_0^{\tau/n} e^{\|A\|\tau/n}\,ds \le (\tau/n)e^{\|A\|\tau/n}$. This yields the desired bound c_1 which is independent of n. □

Existence of Preadjoints

Given an operator H: $X^* \to X^*$, where X^* is the dual of some Banach space X, its preadjoint $^*H$ is such that $^*H$: $X \to X$ and $(^*H)^* = H$. Not every operator has a preadjoint, but the operators that we are dealing with do. For example, $\hat{B}_1$: $L^\infty[0,\tau] \to \mathbb{R}^{n_x}$ has a preadjoint $^*\hat{B}_1$: $\mathbb{R}^{n_x} \to L^1[0,\tau]$. Let $\hat{B}_1(s)$ denote the matrix valued kernel function representing the operator $\hat{B}_1$, it is very easy to check that the operator from $\mathbb{R}^{n_x}$ to $L^1[0,\tau]$ given by the matrix valued kernel function $\hat{B}_1'(t)$ (here $'$ denotes matrix transpose) is a preadjoint to the operator $\hat{B}_1$.

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Bassam A. Bamieh (S'87–M'91) was born in Jordan in 1963. He received the B.S. degree in electrical engineering from Valparaiso University, Valparaiso, IN, in 1983, and the M.Sc. and Ph.D. degrees from Rice University, Houston, TX, in 1986 and 1992, respectively.

He is currently an Assistant Professor in the Department of Electrical and Computer Engineering and in the Coordinated Science Laboratory at the University of Illinois, Urbana-Champaign. His general area of interest is control and system theory; his current areas of interest include robust control, infinite-dimensional systems, and digital control and sampled-data systems.

Munther A. Dahleh (S'84–M'87) was born in August 1962. He received the B.S.E.E. degree from Texas A & M University, College Station, TX in 1983, and the Ph.D. degree from Rice University, Houston, TX 1987.

Since 1987, he has been with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA, where he is currently an Associate Professor. In July 1992 he was awarded the Finmeccanica career development chair. He has held consulting positions with NASA and the C.S. Draper Laboratory since 1988, and with FIAT Research Center since 1991. His current interests include robust control, identification of uncertain systems and adaptation, and control of time-varying and infinite-dimensional systems.

Dr. Dahleh is currently serving as an Associate Editor for the *Systems and Control Letters*. He is a receipient of the 1993 Eckman award.

J. Boyd Pearson, Jr., (S'57–M'58–M'65–SM'77–F'79) received the Ph.D. degree in 1962 from Purdue University, Lafayette, IN.

He served as Assistant Professor of Electrical Engineering at Purdue from 1962 to 1965, when he joined Rice University, Houston, TX, where he is now a J.S. Abercrombie Professor.

Dr. Pearson served as President of the IEEE Control Systems Society in 1984 and received outstanding paper awards in 1985 with Bor-Chin Chang and in 1989 with Munther Dahleh.

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Analysis and design for robust performance with structured uncertainty

Mustafa Khammash *

Electrical Engineering and Computer Engineering Department, Iowa State University, Ames, IA 50011, USA

J.B. Pearson **

Department of Electrical and Computer Engineering, Rice University, P.O. Box 1892, Houston, TX 77251-1892, USA

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Abstract: Necessary and sufficient conditions for stability and performance robustness of discrete-time systems are provided in terms of the spectral radius of a certain nonnegative matrix. The conditions are easily computable and provide a simple and efficient method for computation of the robustness conditions for SISO as well MIMO perturbations. The problem of robust controller synthesis is explored, and an iteration scheme for controller synthesis is introduced.

Keywords: Robustness; robust stability; robust performance; structured uncertainty.

1. Introduction

In the last decade, the control of uncertain systems has gained considerable attention. While system behavior is governed by precise and fixed laws and principles, it is almost always impossible to get an exact mathematical model for the system due to the complexity of such systems and the difficulty of measuring various system parameters and accounting for all its dynamics. As a result, system models which partly capture system behavior must be used, and the system to be controlled is viewed as being uncertain, beyond the information provided by its model.

Aside from system uncertainty, one is usually concerned with signal uncertainty. Uncertain signals model various disturbances which may affect the system. Such disturbances are not completely arbitrary and are usually assumed to belong to a certain class of signals. One such class of signals contains those signals which have a bounded L^2 norm. These signals are bounded in energy. When designing controllers with the objective of minimizing the effect of these signals on the energy of certain system output signals, the H^∞ techniques provide a systematic procedure for achieving this task. In many occasions, however, it is the magnitude of the disturbance and output signals that is of concern. In this case, the class of uncertain signals considered is that containing signals with a bounded L^∞ norm. In this case, the design techniques are provided by the L^1 theory.

While the H^∞ and L^1 methodologies provide analysis and synthesis techniques for nominal linear time-invariant systems, they do not directly address system uncertainty. The issue of system robustness to uncertainty in the H^∞ setup has been addressed by various researchers. Of particular interest is the notion of structured uncertainty. The significance of treating structured uncertainty is that it reduces the conservatism in the analysis and design by incorporating information about the location of the uncertainty in the system. This problem has been introduced and first studied in [13,14,3,15,4]. A robustness measure, termed k_m, which treats structured uncertainty was introduced in [15] and a similar measure, termed the Structured Singular Value (SSV) or μ, was introduced in [3]. The computation of the SSV, which is equal to $1/k_m$, can however be computationally difficult especially in the presence of a large number of perturbation blocks. In particular, exact computation of the SSV is in general possible only when 3 or fewer perturba-

Correspondence to: Dr. M. Khammash, Electrical Engineering and Computer Engineering Department, Iowa State University, Ames, IA 50011, USA.

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tion blocks are present. Computable upper and lower bounds can be used to obtain estimates in the case of more than 3 blocks.

Robustness in the L^1 framework in the presence of norm-bounded perturbations has also been addressed. In [1] necessary and sufficient conditions for stability robustness were provided in the presence of unstructured perturbations, i.e. for one perturbation block. In [6,7] the problem of minimizing, in the presence of unstructured uncertainty, the worst case norm of the sensitivity function subject to stability was addressed, thus adding a performance requirement. An exact expression for the worst case norm of the sensitivity function was provided, and it was shown how controllers which minimize this expression can be designed while achieving robust stability. These results were generalized in [8,7] where necessary and sufficient conditions were derived for stability robustness in the presence of structured norm-bounded uncertainty, and therefore an arbitrary number of perturbation blocks can be treated. These conditions were given in terms of the region in which a system of inequalities has its solution. The system of inequalities is completely determined by the interconnection of the nominal system at hand and stabilizing controller. Even though conditions for stability robustness are important in their own right, they also give conditions for performance robustness. This has been demonstrated in [8] where it was shown that a performance robustness problem can be converted to a stability robustness problem by adding a fictitious perturbation block to represent the performance. The conditions for stability robustness which result are exactly those for performance robustness for the original problem. In this paper we establish a connection between the conditions for stability robustness and the spectral radius of a certain nonnegative matrix. The use of the spectral radius conditions allows one not only to obtain numerically efficient ways for determining when a certain system achieves robust stability and performance, but it also provides the means to design robust controllers for any number of perturbation blocks. This is achieved by an iterative scheme for controller synthesis similar to the $D-K$ iteration in the μ theory. The results in this paper have been previously presented by the authors, without any of the proofs, in [10,9].

2. Notation and preliminaries

We use $\mathbb{R}^+$ to denote nonnegative real numbers.

Since we will be working with signals with bounded magnitude, the class of signals of interest will be ℓ^∞. It is the space of all bounded sequences of real numbers, i.e. $x = \{x(k)\}_{k=0}^\infty \in \ell^\infty$ if and only if $\sup_k |x(k)| < \infty$. If x belongs to ℓ^∞ then

$$\| x \|_\infty = \sup_k | x(k) |.$$

We will be dealing with vector signals with each component belonging to ℓ^∞. The class of such signals will be denoted by ℓ_n^∞ where n is the number of components. Given $x = (x_1, \dots, x_n) \in \ell_n^\infty$,

$$\| x \|_\infty = \max_i \| x_i \|_\infty .$$

Given a stable linear shift-invariant system (LSI), its impulse response will belong to ℓ^1, the space of absolutely summable sequences. If $x \in \ell^1$ then

$$\| x \|_1 = \sum_{k=0}^{\infty} | x(k) | < \infty.$$

The $\mathscr{A}$ norm, $\| \cdot \|_{\mathscr{A}}$, of a z-transform of an ℓ^1 sequence, is the ℓ^1 norm of that sequence. So for an LSI system, this will be the ℓ^1 norm of the impulse response of that system. This is a measure of the maximum amplitude gain of the system or the induced ℓ^∞ norm of that system. For a system matrix, the $\mathscr{A}$-norm is the maximum row sum of individual SISO entry norms. Inthis paper, whenever the $\mathscr{A}$ norm is used, it is assumed that the quantity whose norm is taken is the transfer function of a certain LSI system.

We will be dealing with perturbations of bounded norm. We use $\Delta^{p \times q}$ to denote the set of admissible perturbations. In the set of all operators mapping ℓ_q^∞ to ℓ_p^∞, with induced ℓ^∞ norm less than or equal to one. Hence,

$$\Delta^{p \times q} := \left\{ \Delta : \Delta \text{ is strictly causal and } \sup_{x \neq 0} \frac{\| \Delta x \|_\infty}{\| x \|_\infty} \le 1 \right\}$$

If $p = q = 1$ we will refer to this set as $\boldsymbol{\Delta}$. A related set is

$$\mathscr{D}([(p_1, q_1), \ldots, (p_n, q_n)]),$$

the set of all diagonal operators of the form $\Delta = \text{diag}(\Delta_1, \ldots, \Delta_n)$ where $\Delta_i \in \boldsymbol{\Delta}^{p_i \times q_i}$. If $p_i = q_i = 1$ for all i, we refer to this set as $\mathscr{D}(n)$. It should be noted here that the perturbations are allowed to be time-varying and nonlinear. The results of this paper will remain true if the perturbations are restricted to be linear and possibly time-varying, or if the perturbations are restricted to be time-invariant but allowed to be nonlinear.

For the remainder of this section we collect some of the results available on nonnegative matrices which will be used later on in this paper. A matrix A is said to be nonnegative if all its entries are nonnegative. In this case, we write $A \geq 0$. By $A \geq B$, we mean $A - B \geq 0$. We now state the following definition.

Definition 1. An $n \times n$ matrix A is reducible if there exists an $n \times n$ permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} is an $r \times r$ submatrix and A_{22} is an $n - r \times n - r$ submatrix, with $r < n$.

A matrix which is not reducible is called irreducible. Irreducible nonnegative matrices have a variety of interesting properties. Among these is the following theorem which is a consequences of Perron–Frobenius theory for nonnegative matrices (See [5])

Theorem 1. *Let $A = (a_{ij})$ be a nonnegative square matrix. Then*

⟨1⟩ *A has a nonnegative real eigenvalue equal to its spectral radius, $\rho(A)$.*

⟨2⟩ *To $\rho(A)$ there corresponds an eigenvector $\bar{r} \geq 0$. If A is irreducible then $\bar{r} > 0$.*

⟨3⟩ *$\rho(A)$ is a monotone increasing function of any of the entries of A. If A is irreducible, the monotonicity is strict.*

⟨4⟩ *Let $r_i > 0$ for $i = 1, \ldots, n$. Then*

$$\rho(A) \leq \max_i \frac{1}{r_i} \sum_{j=1}^{n} r_j a_{ij}.$$

⟨5⟩ $$\min_i \sum_{j=1}^{n} a_{ij} \leq \rho(A) \leq \max_i \sum_{j=1}^{n} a_{ij}.$$

3. Problem setup

We start by setting up the robustness problem for LSI systems. Consider the system in Figure 1. In the figure, $\mathscr{G}_0$ is a nominal LSI plant. $\mathscr{C}$ is a LSI controller stabilizing $\mathscr{G}_0$. For the analysis problem, $\mathscr{C}$ is assumed given and fixed. M represents the nominal part of the systems composed of the nominal LSI plant and the LSI stabilizing controller. M can be either continuous-time or discrete-time. We will assume it is discrete-time, although the results carry over with obvious modifications to the continuous-time case. w represents the unknown disturbances. The only assumption on w is that it is bounded in magnitude, i.e. w belongs to the space ℓ^∞, and we assume that $\|w\|_\infty \leq 1$. z, on the other hand, is the regulated output of interest. $\Delta_1, \ldots, \Delta_n$ are the system perturbations modelling the uncertainty. Each perturbation block Δ_i is causal and has an induced ℓ^∞-norm less than or equal to 1. Therefore, each Δ_i belongs to $\boldsymbol{\Delta}^{p_i \times q_i}$. Whereas M is given and fixed (at least in the analysis problem where a controller is given), each perturbation

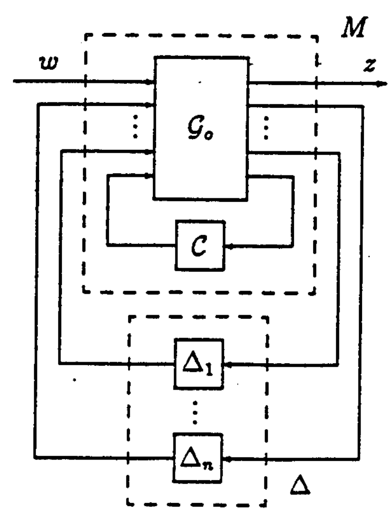

Fig. 1. System with structured uncertainty.

block. Δ_i is allowed to vary over the set $\Delta^{p_i \times q_i}$. The combined effect of all perturbation blocks can be equivalently captured by one perturbation block, Δ, which has a diagonal structure. Δ now belongs to the class

$$\mathscr{D}([(p_1, q_1), \ldots, (p_n, q_n)]).$$

With this setup in mind, the system is said to achieve robust stability if it is ℓ^∞-stable for all admissible perturbations, i.e. for all

$$\Delta \in \mathscr{D}([(p_1, q_1), \ldots, (p_n, q_n)]).$$

It is said to achieve robust performance if it achieves robust stability and at the same time $\| \mathscr{T}_{zw} \| < 1$ for all admissible perturbations, where $\mathscr{T}_{zw}$ is the map between the input w and the output z. Note that when the perturbation Δ is zero, $\| \mathscr{T}_{zw} \|$ is the induced ℓ^∞ norm of the nominal system and is equal to the ℓ^1 norm of the impulse response of the map $\mathscr{T}_{zw}$.

Next, we briefly discuss the relationship that exists between stability robustness and performance robustness. When M is time-invariant, it was shown in [8] that a certain interesting equivalence between stability robustness and performance robustness holds. More specifically, a performance robustness problem can be treated as a stability robustness problem. Consider the two systems in Figure 2. The first system in the figure, System I, is that corresponding to the robust performance problem and has $n-1$ perturbation blocks. By adding a fictitious perturbation block, Δ_P, where $\Delta_P \in \Delta$, we get System II. System II, therefore, corresponds to a stability robustness problem with n perturbation blocks. The following theorem, whose proof can be found in [8], establishes the relation between the two systems:

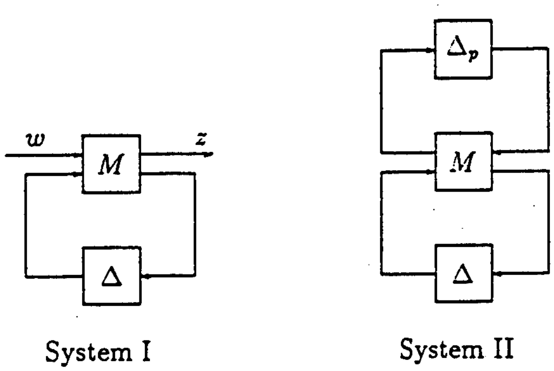

Fig. 2. Stability robustness vs. performance robustness.

Theorem 2. *System* I *achieves robust performance if and only if System* II *achieves robust stability.*

A similar relationship holds when the perturbations are time-invariant and when $w \in \ell^2$ as has been shown in [4]. As a result of this theorem, we can focus our efforts on finding stability robustness conditions. Conditions for performance robustness will automatically follow from the stability conditions.

4. Stability robustness analysis

In this section, we state our main theorem establishing the necessary and sufficient conditions for robust stability of System II in Figure 2 in terms of the spectral radius of a certain matrix as well as other conditions. We do this in the SISO and MIMO cases.

4.1. SISO perturbations

When the perturbations are SISO, the class of admissible perturbations is $\mathscr{D}(n)$. As a result, M will have n inputs and n outputs. It can be expressed as follows:

$$M := \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{n1} & \cdots & M_{nn} \end{bmatrix}.$$

Since M is LSI and stable, each M_{ij} has a pulse response which belongs to the space ℓ^1. The ℓ^1 norm, or the $\mathscr{A}$ norm of M_{ij} can be computed arbitrarily accurately. We define $\hat{M}$ to be the following matrix of norms

$$\hat{M} := \begin{bmatrix} \| M_{11} \|_{\mathscr{A}} & \cdots & \| M_{1n} \|_{\mathscr{A}} \\ \vdots & & \vdots \\ \| M_{n1} \|_{\mathscr{A}} & \cdots & \| M_{nn} \|_{\mathscr{A}} \end{bmatrix}.$$

Defining $\mathscr{R}$ to be the set of all $n \times n$ real diagonal matrices with positive entries on the diagonal, we can state the following theorem:

Theorem 3. *The following are all equivalent:*

⟨1⟩ *System* II *in Figure* 2, *with* $\mathscr{D}(n)$ *as the perturbation class, achieves robust stability.*

⟨2⟩ *The system of inequalities:* $x \le \hat{M}x$ *has no nonzero solution* $x \in \mathbb{R}^n$ *which satisfies* $x \ge 0$.

⟨3⟩ $\rho(\hat{M}) < 1$.
⟨4⟩ $\inf_{R\in\mathscr{R}} \| R^{-1}MR \|_{\mathscr{A}} < 1$.

Proof. That ⟨1⟩ and ⟨2⟩ are equivalent has been shown in [8]. To show ⟨2⟩ implies ⟨3⟩ suppose $\rho(\hat{M}) \geq 1$. Since M is a nonnegative matrix, its spectral radius is itself an eigenvalue and the associated eigenvector, say x, satisfies $x \geq 0$. The fact that $\rho(\hat{M})$ is an eigenvalue implies that $\rho(\hat{M})x = \hat{M}x$. Since $x \geq 0$ and $\rho(\hat{M}) \geq 1$ we have $x \leq \hat{M}x$. Thus, ⟨2⟩ implies ⟨3⟩.

We now show ⟨3⟩ implies ⟨4⟩. Suppose ⟨3⟩ holds. If $\hat{M}$ is reducible, form $\hat{M}_\varepsilon$ by perturbing $\hat{M}$ slightly so that

(a) $\hat{M}_\varepsilon \geq \hat{M}$,
(b) $\hat{M}_\varepsilon$ is irreducible, and
(c) $\rho(\hat{M}_\varepsilon) < 1$.

This can always be done by slightly increasing the zero entries in $\hat{M}$, and using the continuity of the spectral radius to ensure the spectral radius does not increase beyond one. The resulting $\hat{M}_\varepsilon$, being positive, will be irreducible (see [5]). If $\bar{r} > 0$ is the positive eigenvector corresponding to the spectral radius of $\hat{M}_\varepsilon$ guaranteed by Theorem 1 then clearly

$$\rho(\hat{M}_\varepsilon) = \frac{1}{\bar{r}_i} \sum_{j=1}^{n} \bar{r}_j (\hat{M}_\varepsilon)_{ij} \quad \forall i.$$

From Theorem 1, this implies that

$$\rho(\hat{M}_\varepsilon) = \inf_{r_i>0} \max_i \frac{1}{r_i} \sum_{j=1}^{n} r_j (\hat{M}_\varepsilon)_{ij}. \tag{1}$$

Furthermore, $\hat{M}_\varepsilon \geq \hat{M}$ implies that

$$\inf_{r_i>0} \max_i \frac{1}{r_i} \sum_{j=1}^{n} r_j (\hat{M}_\varepsilon)_{ij} \geq \inf_{r_i>0} \max_i \frac{1}{r_i} \sum_{j=1}^{n} r_j \hat{M}_{ij}. \tag{2}$$

Combining (1), (2), and the fact that

$$\| M \|_{\mathscr{A}} = \max_i \sum_{j=1}^{n} \| M_{ij} \|_{\mathscr{A}}$$

we have that

$$\inf_{R\in\mathscr{R}} \| R^{-1}MR \|_{\mathscr{A}} < 1.$$

This completes the proof that ⟨3⟩ implies ⟨4⟩.

Finally, we show that ⟨4⟩ implies ⟨1⟩. Suppose ⟨4⟩ is true. Then for some $R \in \mathscr{R}$,

$$\| R^{-1}MR \|_{\mathscr{A}} < 1.$$

From the Small Gain Theorem, it follows that $(I - R^{-1}MR\Delta)^{-1}$ is ℓ^∞-stable for all $\Delta \in \mathscr{D}(n)$. But since $R^{-1}\Delta R = \Delta$, it follows that

$$(I - R^{-1}MRR^{-1}\Delta R)^{-1}$$

is ℓ^∞-stable for all $\Delta \in \mathscr{D}(n)$. This in turn implies that $R^{-1}(I - M\Delta)^{-1}R$ is ℓ^∞-stable for all $\Delta \in \mathscr{D}(n)$, which is equivalent to $(I - M\Delta)^{-1}$ being ℓ^∞-stable for all $\Delta \in \mathscr{D}(n)$. It can be easily seen [8] that this last condition is necessary and sufficient for robust stability of System II in Figure 2. This completes the proof. □

The proof of Theorem 3 above suggests a way to compute the optimal scaling matrix, R, which achieves the infimum when $\hat{M}$ is irreducible. This is summarized in the following corollary:

Corollary 1. *Let $\hat{M}$ be as defined above. If $\hat{M}$ is irreducible, then*

$$\inf_{R\in\mathscr{R}} \| R^{-1}MR \|_{\mathscr{A}} = \| \bar{R}^{-1}M\bar{R} \|_{\mathscr{A}},$$

where $\bar{R} := \mathrm{diag}(\bar{r}_1, \ldots, \bar{r}_n)$, with $(\bar{r}_1, \ldots, \bar{r}_n)^{\mathrm{T}}$ being the eigenvector corresponding to $\rho(\hat{M})$ which is an eigenvalue of $\hat{M}$.

Proof. Follows directly from the arguments used in the proof of Theorem 3. □

Another fortunate consequence of the nonnegativity of $\hat{M}$ is that both the spectral radius and its corresponding eigenvector can be computed very easily using power methods. To see this consider the following theorem (see [18]):

Theorem 4. *Suppose $\hat{M}$ is primitive ($\hat{M}^m > 0$ for some positive integer m). Let $x > 0$ be any n vector. Then*

$$\min_i \frac{(\hat{M}^{m+1}x)_i}{(\hat{M}^m x)_i} \leq \rho(\hat{M}) \leq \max_i \frac{(\hat{M}^{m+1}x)_i}{(\hat{M}^m x)_i}.$$

Furthermore, the upper and lower bounds both converge to $\rho(\hat{M})$ as $m \to \infty$.

This theorem can be used to compute the spectral radius for large matrices by computing the upper bound and the lower bound for the spectral radius which are guaranteed to converge to the spectral radius. Finally, if $\hat{M}$ were not primitive, it can be made so by replacing every zero entry with an $\varepsilon > 0$. Since the spectral radius is a continuous function of the matrix entries, it follows that the solution of this modified problem will approach that of the original one as ε approaches zero.

4.2. MIMO perturbations

We now discuss the case when each of the perturbations can have multiple inputs and outputs. In this case, the class of admissible perturbations is

$$\mathscr{D}[(p_1, q_1);\dots;(p_n, q_n)] := \{\Delta = \operatorname{diag}(\Delta_1, \dots, \Delta_n) \colon \Delta_i \in \mathbf{\Delta}^{p_i, q_i}\}.$$

We define

$$p := (p_1, \dots, p_n) \quad \text{and} \quad q := (q_1, \dots, q_n).$$

M can be partitioned according to the structure of the perturbations. Hence,

$$M = \begin{bmatrix} M_{11} & \dots & M_{1n} \\ \vdots & & \vdots \\ M_{n1} & \dots & M_{nn} \end{bmatrix}.$$

where M_{ij} has p_j inputs and q_i outputs. In order to refer to the rows of M_{ij} we denote the mth row of M_{ij} by $(M_{ij})_m$.

Before we discuss the next theorem giving necessary and sufficient conditions for stability robustness, we need some definitions. First, define

$$\mathscr{K} := \{(k_1, \dots, k_n) \colon k_i \text{ is a positive integer and } 1 \le k_i \le q_i\}.$$

From this definition it is clear that the set $\mathscr{K}$ has exactly $\prod_{i=1}^{n} q_i$ elements. To each $k = (k_1, \dots, k_n) \in \mathscr{K}$ we define:

$$M_k := \begin{bmatrix} (M_{11})_{k_1} & \dots & (M_{1n})_{k_1} \\ \vdots & & \vdots \\ (M_{n1})_{k_n} & \dots & (M_{nn})_{k_n} \end{bmatrix}.$$

Accordingly, $(M_{ij})_{k_i}$ will have p_j inputs and one output. Similar to the SISO case, we define

$$\hat{M}_k := \begin{bmatrix} \|(M_{11})_{k_1}\|_{\mathscr{A}} & \dots & \|(M_{1n})_{k_1}\|_{\mathscr{A}} \\ \vdots & & \vdots \\ \|(M_{n1})_{k_n}\|_{\mathscr{A}} & \dots & \|(M_{nn})_{k_n}\|_{\mathscr{A}} \end{bmatrix}.$$

As has been shown in [8], it is M_k for $k \in \mathscr{K}$ that determine the robust stability of the system. This will be expressed in the next theorem.

Finally, given $R = \operatorname{diag}(r_1, \dots, r_n) \in \mathscr{R}$, and a vector of positive integers $l = (l_1, \dots, l_n)$, we define

$$R_l := \operatorname{diag}(r_1, \dots, r_1, \dots, r_n, \dots, r_n).$$

with r_i repeated l_i times, $i = 1, \dots, n$. Clearly, R_l depends on both R and l.

Before we state the main robustness theorem for the MIMO case, we need the following lemma:

Lemma 1. *Let A_1 and A_2 be $n \times n$ irreducible nonnegative matrices such that for some $R \in \mathscr{R}$:*

⟨1⟩ All the rows of $R^{-1}A_1R$ have equal sum.

⟨2⟩ The last $n-1$ rows of $R^{-1}A_1R$ and $R^{-1}A_2R$ are identical.

⟨3⟩ The sum of the 1st row of $R^{-1}A_2R$ is strictly larger than the sum of the 1st row of $R^{-1}A_1R$, i.e.

$$\sum_{j=1}^{\infty} (R^{-1}A_1R)_{1j} < \sum_{j=1}^{\infty} (R^{-1}A_2R)_{1j}.$$

Then $\rho(A_1) < \rho(A_2)$.

Proof. It follows from part ⟨5⟩ of Theorem 1 that:

$$\sum_{j=1}^{\infty} (R^{-1}A_1R)_{ij} = \rho(R^{-1}A_1R) \le \rho(R^{-1}A_2R) \quad \forall i.$$

Let $\tilde{A}_2$ be any matrix formed from $R^{-1}A_2R$ by retaining the last $n-1$ rows and subtracting positive quantities from the entries of the first row of $R^{-1}A_2R$ so as to make the sum of the 1st row equal to the sum of any of the remaining rows. Applying ⟨5⟩ of Theorem 1 to $\tilde{A}$ we get that

$$\rho(R^{-1}A_1R) = \rho(\tilde{A}).$$

But from part ⟨3⟩ of Theorem 1 we have that

$$\rho(\bar{A}) < \rho(R^{-1}A_2R).$$

Thus

$$\rho(R^{-1}A_1R) < \rho(R^{-1}A_2R)$$

and the result follows from the similarity of A_i and $R^{-1}A_iR$. □

Theorem 5. *The following are all equivalent:*

⟨1⟩ *System* II *in Figure* 2, *with*

$$\mathscr{D}[(p_1, q_1);\ldots;(p_n, q_n)]$$

as the perturbation class, achieves robust stability.

⟨2⟩ *Given any* $k \in \mathscr{K}$, *the system*

$$x \le \hat{M}_k x,$$

has no nonzero solution $x \in \mathbb{R}^n$ *satisfying* $x \ge 0$.

⟨3⟩ *For all* $k \in \mathscr{K}$, $\rho(\hat{M}_k) < 1$.

⟨4⟩ $\inf_{R\in\mathscr{R}} \| R_q^{-1}MR_p \|_{\mathscr{A}} < 1$.

Proof. The equivalence of ⟨1⟩ and ⟨2⟩ was discussed in [8]. The equivalence of ⟨2⟩ and ⟨3⟩ is a direct consequence of the equivalence of ⟨2⟩ and ⟨3⟩ in Theorem 3. It remains to show the equivalence of ⟨3⟩ and ⟨4⟩. In doing that we will make the assumption that for each $k \in \mathscr{K}$, the $n \times n$ matrix $\hat{M}_k$ is irreducible. If this were not the case, it can be made irreducible by adding to each entry a sufficiently small $\varepsilon > 0$. It is not difficult to show that the equivalence of ⟨3⟩ and ⟨4⟩ for the modified irreducible matrix implies the equivalence of ⟨3⟩ and ⟨4⟩ for the original reducible matrix.

To show the equivalence of ⟨3⟩ and ⟨4⟩, first choose $\bar{k} \in \mathscr{K}$ such that

$$\rho(\hat{M}_{\bar{k}}) = \max_{k\in\mathscr{K}} \rho(\hat{M}_k).$$

Now, let $\bar{R} = \operatorname{diag}(\bar{r}_1, \ldots, \bar{r}_n)$ be the matrix formed from the eigenvector corresponding to $\hat{M}_{\bar{k}}$. We therefore have

$$\rho(\hat{M}_{\bar{k}}) = \sum_j \left(\bar{R}^{-1}\hat{M}_{\bar{k}}\bar{R}\right)_{ij} \quad \forall i. \tag{3}$$

In fact, equation (3), together with part ⟨4⟩ of Theorem 1 and the definition of the $\mathscr{A}$-norm imply that

$$\begin{aligned}\rho(\hat{M}_{\bar{k}}) &= \left\| \bar{R}^{-1}M_{\bar{k}}\bar{R}_p \right\|_{\mathscr{A}} \\ &= \inf_{R\in\mathscr{R}} \left\| R^{-1}M_{\bar{k}}R_p \right\|_{\mathscr{A}}.\end{aligned} \tag{4}$$

It follows from equation (4) and the definition of the $\mathscr{A}$-norm that:

$$\begin{aligned}\rho(\hat{M}_{\bar{k}}) &= \left\| \bar{R}^{-1}M_{\bar{k}}\bar{R}_p \right\|_{\mathscr{A}} \\ &\le \left\| R_q^{-1}MR_p \right\|_{\mathscr{A}} \quad \forall R \in \mathscr{R}.\end{aligned}$$

Accordingly, to prove the equivalence of ⟨3⟩ and ⟨4⟩ it is enough to show that

$$\rho(\hat{M}_{\bar{k}}) = \left\| \bar{R}_q^{-1}M\bar{R}_p \right\|_{\mathscr{A}}. \tag{5}$$

Assuming $\rho(\hat{M}_{\bar{k}}) < \| \bar{R}_q^{-1}M\bar{R}_p \|_{\mathscr{A}}$, we next show that this results in a contradiction, and thus (5) must hold. Without loss of generality, we may assume the $\mathscr{A}$-norm in $\| \bar{R}_q^{-1}M\bar{R}_p \|_{\mathscr{A}}$ is achieved at the first row. Defining $k' := (1, \bar{k}_2, \ldots, \bar{k}_n)$, we therefore have

$$\rho(\hat{M}_{\bar{k}}) < \left\| \left(\bar{R}^{-1}M_{k'}\bar{R}_p\right)_1 \right\|_{\mathscr{A}} = \sum_{j=1}^{n} \left(R^{-1}\hat{M}_{k'}R\right)_{1j}. \tag{6}$$

From equations (3) and (6) it is clear that $\bar{R}^{-1}M_{\bar{k}}\bar{R}$ and $\bar{R}^{-1}M_{k'}\bar{R}$ satisfy the hypothesis of Lemma 1. It follows from Lemma 1 that $\rho(\hat{M}_{\bar{k}}) < \rho(\hat{M}_{k'})$, which is clearly contradiction. The proof is now complete. □

As was the case in the SISO perturbations case, the optimal scaling matrices R_p and R_q achieving the infimum can be computed from a certain eigenvector. Here, the eigenvector used is that corresponding to $\rho(\hat{M}_{\bar{k}})$. This is summarized in the following corollary:

Corollary 2. *Let M be the interconnection system matrix, partitioned according to the structure of the perturbations as shown above. Let* $\bar{k}$ *be such that*

$$\rho(\hat{M}_{\bar{k}}) = \max_k \rho(\hat{M}_k).$$

If $\hat{M}_{\bar{k}}$ *is irreducible, then*

$$\inf_{R\in\mathscr{R}} \left\| R_q^{-1}MR_p \right\|_{\mathscr{A}} = \left\| \bar{R}_q^{-1}M\bar{R}_p \right\|_{\mathscr{A}},$$

where $\bar{R} = \operatorname{diag}(\bar{r}_1, \ldots, \bar{r}_n)$, *with* $\operatorname{diag}(\bar{r}_1, \ldots, \bar{r}_n)$ *being the eigenvector corresponding to* $\rho(\hat{M}_{\bar{k}})$.

Proof. Follows from arguments used in the proof of Theorem 5 above.

5. Optimal scalings and robust controller synthesis

In this section we discuss the uses and some of the limitations of the optimal scalings in the synthesis of robust controllers. Since M forms the interconnection of the nominal LSI system and LSI controller it can be put in the following form:

$$M = T_1 - T_2 Q T_3$$

where T_1, T_2, and T_3 are stable and depend only on the nominal plant. Q, is a free parameter to be chosen from the set of all stable rational function and determines the controller according to the Youla parametrization. In the analysis problem, Q is fixed and, as a result, so is M. For synthesis, we will need to find an appropriate Q which results in a controller providing satisfactory robustness properties. The robustness synthesis problem for SISO perturbations can thus be stated as follows:

Find

$$\inf_{Q \text{ stable}} \rho(\hat{M}) = \inf_{Q \text{ stable}} \inf_{R \in \mathscr{R}} \| R^{-1}(T_1 - T_2 Q T_3) R \|_{\mathscr{A}}.$$

The spectral radius is a nonconvex function of Q and so it is not clear how the optimization problem in the left hand side of the equation above can be solved. The optimization problem at the right hand side of the equation involves a norm minimization, and therefore the following iteration scheme can be used:

1. Set $i := 1$, and $R_0 := I$.
2. Set

$$Q_i := \arg \inf_{Q \text{ stable}} \| R_{i-1}^{-1}(T_1 - T_2 Q T_3) R_{i-1} \|_{\mathscr{A}}.$$

3. Set

$$R_i := \arg \inf_{R \in \mathscr{R}} \| R^{-1}(T_1 - T_2 Q_i T_3) R \|_{\mathscr{A}}.$$

4. Set $i := i + 1$. Go to step 2.

The optimization problem in the second step of the iteration involves solving a standard ℓ^1 norm minimization problem. This problem has been discussed in [2,11,12,16,17] and software packages for its solution exist and involve only linear programming. The optimization problem in the third step involves computing the eigenvector of a certain matrix as shown in Corollary 1 and can be solved easily. Furthermore, it is clear that this iteration converges since the infimum values obtained in the consecutive application of steps 2 and 3 will be monotonically decreasing and bounded below by zero. It is also clear that the iteration procedure can be terminated at step 3 whenever a desirable robustness level is achieved as indicated by the value of the infimum at that step. It should be pointed out at this point that this scheme is similar to the $D-K$ iteration in μ theory. One main difference is that the scaling matrices here are constant (i.e. non-dynamic) as opposed to the frequency dependent scaling matrices which arise in the μ case. As a result, the optimal scaling matrices here are much easier to compute. Having mentioned that, it is important to keep in mind the main difference between the two approaches: the type of perturbations considered here are norm-bounded possibly time-varying, as opposed to the norm-bounded time-invariant perturbations considered in μ theory.

Before we end this section, we make some remarks about the convexity properties of the synthesis problem stated above. While

$$\| R^{-1}(T_1 - T_2 Q T_3) R \|_{\mathscr{A}}$$

is not convex in R, when R is replaced by $\exp(X)$ with $X = \operatorname{diag}(x_1, \ldots, x_n)$, where $x_i \in \mathbb{R}$ then

$$\| \exp(-X) M \exp(X) \|_{\mathscr{A}}$$

will be convex in X. This is a direct consequence of the definition of the $\mathscr{A}$ norm and the convexity of $\exp(\cdot)$. This fact is not used anywhere since the optimum eigenvector can be computed directly by computing the eigenvector corresponding to the spectral radius. It is easy to show that

$$\| R^{-1}(T_1 - T_2 Q T_3) R \|_{\mathscr{A}}$$

is convex in Q when R is fixed. Unfortunately, it is not convex in both R and Q, and one cannot conclude that a local minimizer for this problem is a global one. In fact there are no guarantees that the iteration converges to a local minimum as it may get stuck at a saddle point. Numerical experiments show that the iteration scheme can significantly reduce the spectral radius for many

problems, resulting in a controller with satisfactory robustness levels. At the same time, there are examples for which the final iteration limit is not small enough, and other initial scaling matrices give much better results. In the worst case, the above iteration scheme can be effectively used as a starting point to get solutions which can be further refined using other techniques. This remains an interesting topic for future research.

6. Conclusion

We have shown that certain robustness conditions obtained by the authors in a previous work are closely related to the spectral properties of certain matrices. This sheds a new light on the robustness analysis problem with structured uncertainty, provides new and more efficient methods for the computation of the robustness conditions, and provides new directions for exploring the robust controller synthesis problem.

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The $\mathscr{H}^2$ problem for sampled-data systems *

Bassam Bamieh

Electrical and Computer Engineering and Coordinated Science Laboratory, University of Illinois at Urbana – Champaign, Urbana, IL, USA

J. Boyd Pearson

Department of Electrical and Computer Engineering, Rice University, Houston, TX, USA

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Abstract: In this paper we pose and give a complete solution to an analog of the $\mathscr{H}^2$ problem for sampled-data system. We motivate and develop a natural generalization of the $\mathscr{H}^2$ cost to periodic systems, which is then applied to the continuous-time closed-loop mapping in a sampled-data control system. It is shown that the cost criterion developed is actually a norm in an $\mathscr{H}^2$ space of Hilbert–Schmidt operator valued functions. We give state space solutions to the optimal and suboptimal controllers synthesis problems in this new norm by establishing an equivalent standard $\mathscr{H}^2$ problem.

Keywords: $\mathscr{H}^2$-optimization; sampled-data systems; periodic systems; lifting technique; infinite dimensional systems.

Introduction

We consider control systems made up of a continuous-time time-invariant generalized plant and a discrete-time time-invariant controller connected together in feedback by sample and hold devices. Figure 1 shows this arrangement which is a sampled-data control version of the so-called 'standard problem'. We call the mapping from the exogenous input w to the regulated output z, the closed-loop mapping T_{wz}. One generally desires to synthesize a controller such that some norm of the closed-loop mapping is minimized or kept small. When T_{wz} is time invariant, the more popular norms to minimize are the L^1, $\mathscr{H}^\infty$ or the $\mathscr{H}^2$ norms.

In the sampled-data system of Figure 1, T_{wz} is not time invariant but is periodically time varying due to the presense of the sample and hold devices. It is necessary to deal with sampled-data systems as time

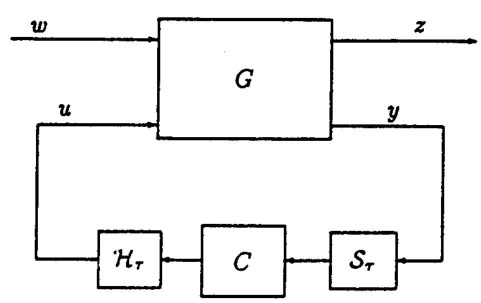

Fig. 1. Hybrid discrete/continuous-time system.

Correspondence to: B. Bamieh, Electrical and Computer Engineering and Coordinated Science Laboratory, University of Illinois, 1101 West Springfield Avenue, Urbana, IL 61801, USA.

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varying, if one is to consider their continuous time (e.g. inter-sample) behavior, to emphasize this fact we call the system in Figure 1 a *hybrid* system. In the case of the L^1 or the $\mathscr{H}^\infty$ problems, there is little ambiguity in generalizing the problem statement to include time-varying systems. The L^1 and $\mathscr{H}^\infty$ problems have been originally motivated as those of minimizing the L^∞ and L^2 induced norms (respectively) of systems, and it is natural to pose the problem of minimizing these induced norms for time-varying systems.

The $\mathscr{H}^2$ problem, on the other hand, is normally stated for time-invariant systems, and there is no immediate or widely accepted generalization of this norm to time-varying systems. There are several cost criteria which can be proposed that are equivalent to the $\mathscr{H}^2$ cost when specialized to time-invariant systems, but are different when applied to time-varying systems. Recently, [3] considered a problem where the L^2 norm of the response to a delta input is minimized, and [2] considered problems similar to minimizing the L^2 to L^∞ and the L^1 to L^2 induced norms respectively.

In the following section, we will present a different generalization of the $\mathscr{H}^2$ cost which we will argue is the natural one. We review the deterministic set up of the $\mathscr{H}^2$ problem and from it motivate a generalized deterministic time-varying problem, then we derive an expression for the generalized cost for periodic systems in terms of their lifted equivalents. We then recall the stochastic interpretation of the $\mathscr{H}^2$ problem and show that the proposed generalization has a natural stochastic interpretation as well. In the second section, we solve this new $\mathscr{H}^2$ problem for sampled-data systems by establishing an equivalence between it and a standard discrete-time time-invariant $\mathscr{H}^2$ problem [1].

Sampled-data systems are periodically time varying in continuous time. The analysis of periodic systems is greatly simplified by the use of the *lifting* technique, which provides for a strong equivalence between continuous-time periodic systems and a certain type of infinite dimensional but time-invariant systems. Now we recall very briefly some facts about the lifting framework from [1], [2], we refer the reader to these papers for the full details. A continuous-time τ-periodic L^2-stable linear system G can be put in correspondence with a discrete-time shift-invariant system $\hat{G}$ which acts on $L^2[0, \tau]$-valued discrete-time signals, i.e. signals in the space $\ell^2_{L^2[0,\tau]}$. $\hat{G}$ is called the *lifting* of G, and it has infinite-dimensional input and output spaces but a finite dimensional state space (of the same dimension as G); for lack of a better term, we will abuse terminology by calling such systems infinite-dimensional. The system $\hat{G}$ is represented by a convolution sum in terms of what might be called its operator valued 'impulse response' sequence $\{\hat{G}_i\}$. For each i, $\hat{G}_i$ is an operator on $L^2[0, \tau]$, and the system $\hat{G}$ acts on a signal $\{\hat{u}_i\} \in \ell^2_{L^2[0,\tau]}$ by

$$\hat{y} = \hat{G}\hat{u}, \qquad \hat{y}_i = \sum_{j=0}^{i} \hat{G}_{i-j}\hat{u}_j.$$

The generalized $\mathscr{H}^2$ cost we define will take on a very familiar form when viewed in terms of the lifted system $\hat{G}$.

1. Generalizing the $\mathscr{H}^2$ cost to time varying systems

1.1. Deterministic setup

Let us begin by stating the $\mathscr{H}^2$ problem for scalar time-invariant systems. Let $G(t)$ be the impulse response of a scalar time-invariant system (strictly proper, we assume this from now on) which we donate by G. The $\mathscr{H}^2$ norm of the system is defined by

$$\|G\|^2_{\mathscr{H}^2} := \int_0^\infty (G(t))^2 \, \mathrm{d}t. \tag{1}$$

[1] After this work was completed, we received [5], where the same problem was considered.

This norm is usually interpreted deterministically as the norm of the response to a fixed input. If by $G[u]$ we denote the output of the system given input u, then we have

$$\|G\|_{\mathscr{H}^2}^2 = \|G[\delta]\|_{L^2}^2,$$

where the input δ denotes the delta generalized function $\delta(t)$, and L^2 is the Hilbert space $L^2[0, \infty)$. Thus the $\mathscr{H}^2$ norm of the system G is the L^2 norm of its response to a single input. This interpretation of the $\mathscr{H}^2$ norm breaks down in the multivariable case. If G is a multivariable time-invariant system and $G(t)$ is its matrix-valued impulse response, the $\mathscr{H}^2$ norm is defined by

$$\|G\|_{\mathscr{H}^2}^2 := \operatorname{tr}\left(\int_0^\infty G'(t)G(t)\, \mathrm{d}t\right). \tag{2}$$

The norm so defined cannot be given the interpretation of the L^2 norm of the response to a fixed input. However, a slightly different interpretation can be given as follows; if by δ^i we mean the vector signal which has a delta function in the i-th position and zeros in the other positions, then the definition given in (2) is equivalent to

$$\|G\|_{\mathscr{H}^2}^2 = \sum_{i=1}^{n} \|G[\delta^i]\|_{L^2}^2, \tag{3}$$

where n is the number of inputs of G. So the $\mathscr{H}^2$ norm is the sum of the squares of the L^2 norms of the responses to n different inputs; it will be more useful for us to think of this sum as a square average. The set of n inputs 'excite' every input channel of the system, and then a square average is taken of all the norms of the different responses. Note that (3) does not exactly represent an average since a factor of $1/n$ is missing, thus it is actually a multiple of the average, but since we are only interested in motivating an interpretation of the $\mathscr{H}^2$ norm, we disregard this difference. If we had used a single input such as $x\delta(t)$, where x is some vector in $\mathbb{R}^n$, then this input excites the system in only one 'direction', namely that of x. Thus, (3) characterizes the $\mathscr{H}^2$ norm as the *square average of the norms of the responses to a certain set of inputs*, where the set of inputs are chosen so as to excite all 'parts' of the system.

The generalization of the $\mathscr{H}^2$ cost which we will give is motivated by the interpretation just given for the $\mathscr{H}^2$ cost of a multivariable system. We start with the case of periodic systems. Consider a scalar but τ-periodic system G described by its kernel (time-varying impulse response) $G(t, s)$, which is a doubly-periodic function in t and s. The response of G to the single input $\delta(t)$ is given by $G(t, 0)$. Since G is not time invariant, this response $G(t, 0)$ could be very different from the response to $\delta_h(t) := \delta(t-h)$, a delta applied at some other time h. Since the system is τ-periodic we can think of applying many different inputs δ_h to excite the different 'parts' of the kernel $G(t, s)$. Since $G(t, s)$ is a τ-periodic function of s, it is completely determined by its responses to the inputs δ_h, for $0 \leqslant h \leqslant \tau$. So by analogy with the multivariable time invariant case, our set of inputs for a τ-periodic system is $\{\delta_h;\ 0 \leqslant h \leqslant \tau\}$, and the generalized $\mathscr{H}^2$ cost should be the square 'average' of the L^2 norms of the responses to inputs in this set. Formally, given a scalar τ-periodic system G, its $\mathscr{H}^2$ norm is defined by

$$\|G\|_{\mathscr{H}^2}^2 := \frac{1}{\tau}\int_0^\tau \|G[\delta_h]\|_{L^2}^2\, \mathrm{d}h.$$

Thus the 'average' is taken by integration. Note how this definition reduces to the standard $\mathscr{H}^2$ norm if the system is time invariant, since $\|G[\delta_h]\|_{L^2}$ is a constant function of h if G is time invariant. In terms of the kernel function $G(t, s)$ we have

$$\|G\|_{\mathscr{H}^2}^2 = \frac{1}{\tau}\int_0^\tau \|G(\cdot, h)\|_{L^2}^2\, \mathrm{d}h = \frac{1}{\tau}\int_0^\tau \left(\int_0^\infty (G(t, h))^2\, \mathrm{d}t\right) \mathrm{d}h.$$

For a multivariable system G, the appropriate definition is then

$$\|G\|_{\mathscr{H}^2}^2 := \frac{1}{\tau}\int_0^\tau \operatorname{tr}\left(\int_0^\infty G'(t, h)G(t, h)\, \mathrm{d}t\right) \mathrm{d}h. \tag{4}$$

1.2. The $\mathscr{H}^2$ norm for periodic systems

The definition given earlier in (4) takes on a form very similar to that for time-invariant systems when viewed in terms of the *lifting* of periodic systems. In fact, we will show that the generalized cost is a norm in an $\mathscr{H}^2$ space of operator valued functions.

To set this up, we first need to review some facts about Hilbert–Schmidt operators. The theory of Hilbert–Schmidt operators generalizes notions related to the Frobenius norm for matrices. The space of $n \times m$ matrices is the linear space of all linear operators from the Hilbert space $\mathbb{R}^n$ to the Hilbert space $\mathbb{R}^m$. We denote it by $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$; as such, it is a Banach space if given the induced norm. $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ can also be viewed as a Hilbert space if given the Frobenius norm

$$\| A \|_{\mathrm{F}}^2 := \sum_{i,j} a_{ij}^2 = \operatorname{tr}(A'A). \tag{5}$$

The inner product corresponding to this norm is given by

$$\langle A, B \rangle := \operatorname{tr}(A'B).$$

With this norm, $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ is isometrically isomorphic to $\mathbb{R}^{n \times m}$. The Frobenius and induced norms are related to the singular values $\{\sigma_i\}$ of a matrix A by

$$\| A \| = \max_i \sigma_i, \qquad \| A \|_{\mathrm{F}}^2 = \operatorname{tr}(A'A) = \sum_i \sigma_i^2, \tag{6}$$

thus both norms are in some sense measures of the 'size' of a matrix in terms of its singular values. In this notation, the $\mathscr{H}^2$ norm of a time-invariant system given by its matrix valued impulse response $\{G(t)\}$ is given by

$$\| G \|_{\mathscr{H}^2}^2 := \operatorname{tr}\left(\int_0^\infty G'(t) G(t) \, \mathrm{d}t \right) = \int_0^\infty \operatorname{tr}(G'(t)G(t)) \, \mathrm{d}t = \int_0^\infty \| G(t) \|_{\mathrm{F}}^2 \, \mathrm{d}t,$$

or if $\{G_i\}$ is the matrix valued impulse response sequence of a discrete-time system, then

$$\| G \|_{\mathscr{H}^2}^2 := \sum_{i=0}^{\infty} \operatorname{tr}(G_i' G_i) = \sum_{i=0}^{\infty} \| G_i \|_{\mathrm{F}}^2 = \frac{1}{2\pi} \oint \operatorname{tr}(G'(z)G(z)) \, \mathrm{d}z = \frac{1}{2\pi} \oint \| G(z) \|_{\mathrm{F}} \, \mathrm{d}z, \tag{7}$$

where the last two equations are in terms of the z-transform of G, denoted by $G(z)$.

Now let H_1, H_2 be two Hilbert spaces, the space $\mathscr{L}(H_1, H_2)$ given the induced norm is a Banach space. The class of Hilbert–Schmidt operators is a subspace of $\mathscr{L}(H_1, H_2)$ which can itself be endowed with a Hilbert space structure using a norm which is generally different from the induced norm. Let us consider operators in $\mathscr{L}(L_n^2[0, \tau], L_m^2[0, \tau])$ which can be represented by matrix-valued kernel functions as [2]

$$y = Ku, \quad u \in L_n^2[0, \tau]; \; y \in L_m^2[0, \tau], \qquad y(t) = \int_0^\tau K(t, s) u(s) \, \mathrm{d}s.$$

Such an operator is called Hilbert–Schmidt (HS) if

$$\| K \|_{\mathrm{HS}}^2 := \int_0^\tau \int_0^\tau \operatorname{tr}(K'(t, s) K(t, s)) \, \mathrm{d}t \, \mathrm{d}s < \infty, \tag{8}$$

and the HS norm is given by the above equation. It can be shown that with this norm, the set of HS operators is a Hilbert space. We denote this space by $\mathrm{HS}(L_n^2[0, \tau], L_m^2[0, \tau])$, or simply HS. In our particular application, the operators we will consider on $L^2[0, \tau]$ will be represented by kernel functions

[2] Throughout this paper, we use the same symbol to denote an operator and its kernel function representation, e.g. K and $K(t, s)$ above.

$K(t, s)$ which are bounded on $[0, \tau] \times [0, \tau]$ and thus are immediately HS. The relationship between the Frobenius and the HS norms is made clear by the following facts [4, Example XI.2.20]. Let K^* denote the adjoint operator to K, if K is a HS operator, then K^*K is a compact self adjoint operator, and being compact, it has a countable number of eigenvalues. Furthermore, its eigenvalue sequence $\{\lambda_i(K^*K)\}$ is summable, that is

$$\operatorname{trace}(K^*K) := \sum_{i=1}^{\infty} \lambda_i(K^*K) < \infty,$$

where the 'trace' defined above is called the trace norm. Note that we use 'tr' to denote the trace of a matrix, while 'trace' denotes the trace of an operator. Our use for the notion of a trace of an operator is based on the following relationship between the trace of K^*K and the HS norm of K [4, Example IX.2.20]:

$$\| K \|_{\mathrm{HS}}^2 = \operatorname{trace}(K^*K) = \sum_{i=0}^{\infty} \lambda_i(K^*K). \tag{9}$$

In the space of HS operators, the inner product is given by

$$\langle G_1, G_2 \rangle = \operatorname{trace}(G_1^* G_2)$$

A comparison between (6) and (9) shows the parallels between the Frobenius and the HS norms.

We return now to the generalized $\mathscr{H}^2$ norm for periodic systems. Let $G(t, s)$ be the kernel function of an L^2-stable periodic system. We make the assumption that it is a bounded function on bounded subsets of $\mathbb{R}^2$. Recall that the lifting of G, denoted by $\hat{G}$, is a discrete-time time-invariant system acting on $L^2[0, \tau]$-valued signals. The impulse response of $\hat{G}$ is the operator valued sequence $\{\hat{G}_i\}$, where the kernel representation of each operator is given [1] by

$$\hat{G}_i(\hat{t}, \hat{s}) = G(\hat{t} + i\tau, \hat{s}), \quad \hat{t}, \hat{s} \in [0, \tau].$$

Recall (4) defining the $\mathscr{H}^2$ cost for a periodic system

$$\| G \|_{\mathscr{H}^2}^2 = \frac{1}{\tau} \int_0^{\tau} \operatorname{tr}\left(\int_0^{\infty} G'(t, h) G(t, h) \, \mathrm{d}t \right) \mathrm{d}h.$$

After rearranging the right hand side in terms of the lifted components $\{\hat{G}_i\}$, we get

$$\begin{aligned} \| G \|_{\mathscr{H}^2}^2 &= \frac{1}{\tau} \int_0^{\tau} \operatorname{tr}\left(\sum_{i=0}^{\infty} \int_0^{\tau} G'(\hat{t} + \tau i, h) G(\hat{t} + \tau i, h) \, \mathrm{d}\hat{t} \right) \mathrm{d}h \\ &= \frac{1}{\tau} \int_0^{\tau} \operatorname{tr}\left(\sum_{i=0}^{\infty} \int_0^{\tau} \hat{G}_i'(\hat{t}, h) \hat{G}_i(\hat{t}, h) \, \mathrm{d}\hat{t} \right) \mathrm{d}h \\ &= \frac{1}{\tau} \sum_{i=0}^{\infty} \operatorname{tr}\left(\int_0^{\tau} \left(\int_0^{\tau} \hat{G}_i'(\hat{t}, h) \hat{G}_i(\hat{t}, h) \, \mathrm{d}h \right) \mathrm{d}\hat{t} \right) = \frac{1}{\tau} \sum_{i=0}^{\infty} \| \hat{G}_i \|_{\mathrm{HS}}^2, \end{aligned}$$

$$\| G \|_{\mathscr{H}^2}^2 = \frac{1}{\tau} \sum_{i=0}^{\infty} \operatorname{trace}\left(\hat{G}_i^* \hat{G}_i \right) = \frac{1}{\tau} \sum_{i=0}^{\infty} \| \hat{G}_i \|_{\mathrm{HS}}^2. \tag{10}$$

Note how this formula (10) resembles the definition of the $\mathscr{H}^2$ norm for time-invariant multivariable systems (7) with the Frobenius norm replaced by the HS norm.

This new norm allows us to put a Hilbert space structure on a large class of τ-periodic systems (which includes closed-loop stable sampled data systems). As before, let G be a strictly causal τ-periodic system whose kernel function is a bounded function on bounded subsets of $\mathbb{R}^2$ (the closed-loop operator in a sampled data system satisfies this condition). If $\{\hat{G}_i\}$ is the operator valued 'impulse response' of the

lifting of G, then the previous condition implies that for each i, $\hat{G}_i$ is HS, i.e. $\hat{G}_i \in \mathrm{HS}(L_n^2[0, \tau], L_m^2[0, \tau])$. If we form the space ℓ_{HS}^2 with the norm precisely given by (10) we obtain a Hilbert space. One can take z-transforms of elements in ℓ_{HS}^2 by

$$\hat{G}(z) := \sum_{i=0}^{\infty} z^i \hat{G}_i,$$

and following [7, Chapter 5], the image of ℓ_{HS}^2 under the z-transform is exactly $\mathscr{H}_{\mathrm{HS}}^2$, which is the $\mathscr{H}^2$ space of HS-valued functions that ar analytic in the unit disc with the norm given by

$$\| \hat{G}(z) \|_{\mathscr{H}_{\mathrm{HS}}^2} := \frac{1}{2\pi} \oint \| \hat{G}(z) \|_{\mathrm{HS}} \, \mathrm{d}z = \frac{1}{2\pi} \oint \mathrm{trace}\big(\hat{G}^*(z)\hat{G}(z)\big) \, \mathrm{d}z, \tag{11}$$

where the integral $\oint$ is over the unit circle. The z-transform affords an isometric isomorphism between ℓ_{HS}^2 and $\mathscr{H}_{\mathrm{HS}}^2$, thus this identification justifies calling this new norm an $\mathscr{H}^2$ norm. The striking similarity between the expressions for the $\mathscr{H}_{\mathrm{HS}}^2$ norm (10), (11) for periodic systems, and the standard $\mathscr{H}^2$ norm for time-invariant systems (7), argues that this is the natural extension of the $\mathscr{H}^2$ norm to periodic systems.

1.3. Stochastic interpretation

Let $\{u(t)\}$ be a zero mean stationary white noise stochastic process defined on the time interval $(-\infty, +\infty)$. If $\{u(t)\}$ is the input to a stable linear time-invariant system G, then the output process $y = Gu$, is stationary and the variance at any time is equal to the $\mathscr{H}^2$ norm of G:

$$\mathrm{tr}(E\{ y(t) y'(t)\}) = \| G \|_{\mathscr{H}^2}^2.$$

Thus the $\mathscr{H}^2$ norm is usually given the interpretation as the variance of the error resulting from an input of white noise.

If the input is a white noise process on the time interval $[0, \infty)$, it is no longer stationary and the output is also not stationary, and the variance of the output process depends on time. In this case the $\mathscr{H}^2$ norm is the steady state value of this variance, i.e.,

$$\lim_{t \to \infty} \mathrm{tr}(E\{ y(t) y'(t)\}) = \| G \|_{\mathscr{H}^2}^2.$$

The above expression leads to a possible definition of the generalized $\mathscr{H}^2$ cost when G is time varying. In this case the output process is no longer stationary, and one might think of defining the cost as the 'asymptotic average' of the output variance (AOV) by

$$\mathrm{AOV} := \lim_{M \to \infty} \frac{1}{M} \int_0^M \mathrm{tr}(E\{ y(t) y'(t)\}) \, \mathrm{d}t.$$

We will see that AOV is the generalized $\mathscr{H}^2$ norm previously defined for periodic systems.

Let G be a causal time-varying linear system that is given by its kernel function $G(t, s)$. If the input $\{u(t)\}$ is a white noise process supported on $[0, \infty)$, then the output process y has a correlation function given [6] by

$$E\{y(t_1) y(t_2)\} = R_y(t_1, t_2) = \int_0^{t_1} G(t_1, r) G'(t_2, r) \, \mathrm{d}r.$$

Note that $R_y(t_1, t_2)$ is thus equal to the kernel of the operator GG^*. From the above equation the expression for AOV is

$$\mathrm{AOV} = \lim_{M \to \infty} \frac{1}{M} \int_0^M \mathrm{tr}\big(R_y(t, t)\big) \, \mathrm{d}t = \lim_{M \to \infty} \frac{1}{M} \int_0^M \mathrm{tr}\left(\int_0^t G(t, r) G'(t, r) \, \mathrm{d}r \right) \mathrm{d}t. \tag{12}$$

The above expression can be interpreted in terms of the Hilbert–Schmidt norm defined earlier. For a linear operator $A: L^2[0, \infty) \to L^2[0, \infty)$, denote by $\Pi_M(A)$ the 'truncated' operator $\Pi_M(A) := \Pi_{L^2[0,M]} A \,|\, _{L^2[0,M]}$. If the operator A is represented by a kernel function $A(t, s)$, it then follows that the operator $\Pi_M(A)$ is represented by the 'truncated' kernel function, i.e. $A(t, s)$ for $t, s \in [0, M]$. In light of this fact and the definition of the HS norm (8), (12) can be rewritten as (after adjusting the limit of integration, since G is causal)

$$\mathrm{AOV} = \lim_{M\to\infty} \frac{1}{M} \| \Pi_M(G) \|_{\mathrm{HS}}^2. \tag{13}$$

We note that the above expression is valid whenever the limit exists. One can take equation (13) as a definition of the generalized $\mathscr{H}^2$ cost for time-varying systems. It remains to show that this definition agrees with that used earlier for a τ-periodic systems (see (10)). First note that from the definition of the lifting [1] (after some manipulation) we have,

$$\frac{1}{n\tau} \| \Pi_{n\tau}(G) \|_{\mathrm{HS}}^2 = \frac{1}{\tau}\left[\| \hat{G}_0 \|_{\mathrm{HS}}^2 + \frac{n-1}{n} \| \hat{G}_1 \|_{\mathrm{HS}}^2 + \cdots + \frac{1}{n} \| \hat{G}_{n-1} \|_{\mathrm{HS}}^2 \right], \tag{14}$$

where n is an integer. It is now easy to show that the limit of the above expression (as $n \to \infty$) is precisely $(1/\tau)\sum_{i=0}^{\infty} \| \hat{G}_i \|_{\mathrm{HS}}^2$ (see Appendix), which agrees with the previously defined norm for periodic systems.

Finally we note that in contrast to periodic systems, where the generalized $\mathscr{H}^2$ norm (or AOV) is actually a norm on the subspace of τ-periodic systems for which it is finite, the AOV is only a seminorm on the subspace of time-varying systems for which it is finite.

2. Optimizing the $\mathscr{H}_{\mathrm{HS}}^2$ norm of sampled-data systems

We will now address the problem of optimizing the $\mathscr{H}_{\mathrm{HS}}^2$ norm for sampled-data systems. Since the $\mathscr{H}_{\mathrm{HS}}^2$ norm has a more convenient expression in terms of the lifting of the closed-loop operator, we will work with the lifted equivalent of the sampled-data system. Figure 2 shows the original hybrid system and the equivalent lifted system. We will show that the $\mathscr{H}_{\mathrm{HS}}^2$ norm of the lifted system is equal to the $\mathscr{H}^2$ norm of a certain standard (i.e. finite-dimensional input and output) system, thus converting the $\mathscr{H}_{\mathrm{HS}}^2$ operator-valued problem into a standard matrix-valued $\mathscr{H}^2$ problem.

Let the original generalized plant G (Figure 2(a)) be given by the following state space realization:

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & 0 & 0 \end{array}\right].$$

The assumptions that $D_{21} = D_{22} = 0$ are made to guarantee that the sampler operates on continuous signals. The assumption $D_{11} = 0$ is necessary for the closed-loop operator to be strictly causal. Figure

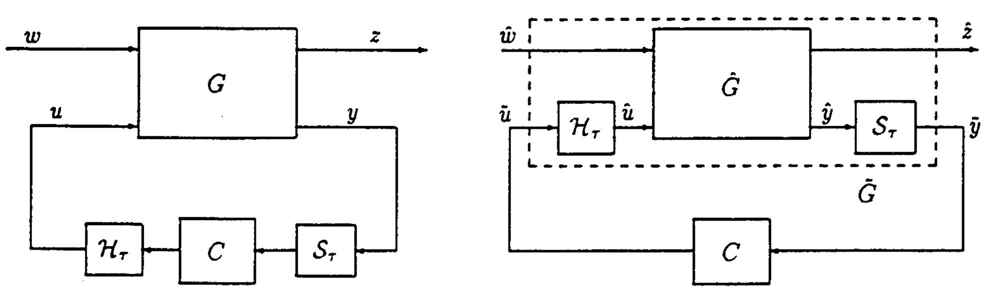

Fig. 2. Left (a): Hybrid system. Right (b): lifted system with discrete-time controller.

2(a) represents the so-called standard problem, where G is the generalized plant which contains the original plant and the systems interconnections. In our setup, the controller in the feedback loop is constrained to be a sampled-data controller, that is, it is in the form $\mathscr{H}_\tau C \mathscr{S}_\tau$, where C is a time-invariant discrete-time system, and $\mathscr{H}_\tau$, $\mathscr{S}_\tau$ are the synchronized hold and sample operators (with period τ) respectively.

Let $T_{wz} =: \mathscr{F}(G, \mathscr{H}_\tau C \mathscr{S}_\tau)$ denote the closed-loop mapping. To work with the $\mathscr{H}^2_{\mathrm{HS}}$ norm defined in the previous section, one needs to obtain expressions for the lifting $\hat{T}_{wz}$ of T_{wz}. This is accomplished as in [1] by lifting the generalized plant G and adjoining the sample and hold operators to yield a new generalized plant $\tilde{G}$ (Figure 2(b)) such that $\mathscr{F}(\tilde{G}, C) = \hat{T}_{wz}$ (see [1] for the details). A discrete-time state space realization of $\tilde{G}$ is given by

$$\tilde{G} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \tilde{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right], \tag{15}$$

with

$$\hat{B}_1 : L^2[0, \tau] \to \mathbb{R}^x, \qquad \hat{A} : \mathbb{R}^x \to \mathbb{R}^x, \qquad \hat{C}_1 : \mathbb{R}^x \to L^2[0, \tau],$$
$$\hat{D}_{11} : L^2[0, \tau] \to L^2[0, \tau], \qquad \tilde{D}_{12} : \mathbb{R}^u \to L^2[0, \tau],$$

where x and u are the dimensions of the state vector and the control input vector respectively, and $\tilde{B}_2$, $\tilde{C}_2$ are finite matrices, and

$$\hat{A} = \mathrm{e}^{A\tau}, \quad \hat{B}_1 = \mathrm{e}^{A(\tau - s)} B_1, \quad \hat{C}_1 = C_1\, \mathrm{e}^{At}, \quad \hat{D}_{11} = C_1\, \mathrm{e}^{A(t-s)} \mathbf{1}_{(t-s)} B_1, \tag{16a}$$

and

$$\tilde{D}_{12} = C_1 \Psi(t) B_2 + D_{12}, \quad \tilde{B}_2 = \Psi(\tau) B_2, \quad \tilde{C}_2 = C_2, \tag{16b}$$

where $\mathbf{1}(\cdot)$ is the unit step function and $\Psi(t) := \int_0^t \mathrm{e}^{As}\, \mathrm{d}s$. Note that the operators are given (where appropriate) in terms of their representing kernel functions.

The problem now is to minimize the $\mathscr{H}^2_{\mathrm{HS}}$ norm of $\mathscr{F}(\tilde{G}, C)$. The next theorem establishes an equivalence between this norm and the standard $\mathscr{H}^2$ norm of $\mathscr{F}(\overline{G}, C)$, where $\overline{G}$ is a standard discrete-time generalized plant constructed from the original problem data, and C is the same controller.

Theorem 1. *Given the infinite dimensional generalized plant $\tilde{G}$ (15), form the finite dimensional generalized plant*

$$\overline{G} = \left[\begin{array}{c|cc} \hat{A} & \overline{B}_1 & \tilde{B}_2 \\ \hline \overline{C}_1 & 0 & \overline{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right],$$

where $\overline{B}_1$, $\overline{C}_1$ and $\overline{D}_{12}$ are finite matrices such that

$$\overline{B}_1 \overline{B}_1' = \hat{B}_1 \hat{B}_1^*, \qquad \begin{bmatrix} \overline{C}_1' \\ \overline{D}_{12}' \end{bmatrix} \begin{bmatrix} \overline{C}_1 & \overline{D}_{12} \end{bmatrix} = \begin{bmatrix} \hat{C}_1^* \\ \tilde{D}_{12}^* \end{bmatrix} \begin{bmatrix} \hat{C}_1 & \tilde{D}_{12} \end{bmatrix}. \tag{17}$$

Then we have

$$\| \mathscr{F}(\tilde{G}, C) \|^2_{\mathscr{H}^2_{\mathrm{HS}}} = \frac{1}{\tau} \left(\| \hat{D}_{11} \|^2_{\mathrm{HS}} + \| \mathscr{F}(\overline{G}, C) \|^2_{\mathscr{H}^2} \right). \tag{18}$$

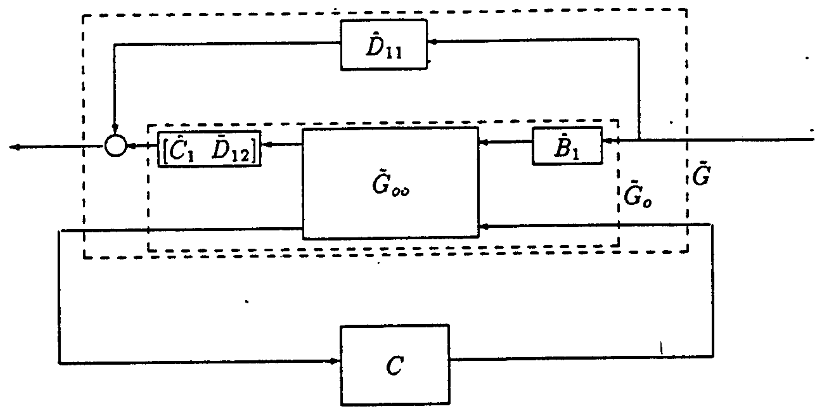

Fig. 3. Decomposition of $\tilde{G}$.

Remark. The operator compositions on the right hand sides of the equations in (17) yield finite matrices; the matrices $\bar{B}_1$, $\bar{C}_1$, $\bar{D}_{12}$ can then be obtained, for example, by Cholesky factorizations, or symmetric factorizations.

Proof. The proof is accomplished by first performing a decomposition of $\tilde{G}$ as follows: First define

$$\tilde{G} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \tilde{B}_2 \\ \hline \hat{C}_1 & 0 & \tilde{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right] + \begin{bmatrix} \hat{D}_{11} & 0 \\ 0 & 0 \end{bmatrix} =: \tilde{G}_0 + \begin{bmatrix} \hat{D}_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

We further decompose $\tilde{G}_0$ into

$$\tilde{G}_0 = \begin{bmatrix} \begin{bmatrix} \hat{C}_1 & \tilde{D}_{12} \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c|cc} \hat{A} & I & \tilde{B}_2 \\ \hline \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \tilde{C}_2 & 0 & 0 \end{array}\right] \begin{bmatrix} \hat{B}_1 & 0 \\ 0 & I \end{bmatrix}$$

$$=: \begin{bmatrix} \begin{bmatrix} \hat{C}_1 & \tilde{D}_{12} \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} \tilde{G}_{00} \begin{bmatrix} \hat{B}_1 & 0 \\ 0 & I \end{bmatrix}.$$

This decomposition is illustrated in Figure 3. Note that with this decomposition, $\tilde{G}_{00}$ is a finite-dimensional system. We also have from Figure 3,

$$\mathscr{F}(\tilde{G}, C) = \hat{D}_{11} + \begin{bmatrix} \hat{C}_1 & \tilde{D}_{12} \end{bmatrix} \mathscr{F}(\tilde{G}_{00}, C) \hat{B}_1.$$

To apply the definition of the $\mathscr{H}^2_{\mathrm{HS}}$ norm, let $\{T_i\}$ denote the operator-valued impulse response of $\mathscr{F}(\tilde{G}, C)$, and $\{(T_{00})_i\}$ the matrix-valued impulse response of $\mathscr{F}(\tilde{G}_{00}, C)$. It is easy to check that the 'direct transmission' term $(T_{00})_0 = 0$. Then we have

$$T_i = \begin{cases} \hat{D}_{11}, & i = 0, \\ \begin{bmatrix} \hat{C}_1 & \tilde{D}_{12} \end{bmatrix} (T_{00})_i \hat{B}_1, & i \geqslant 1. \end{cases}$$

Therefore

$$\begin{aligned}
\|\mathcal{F}(\tilde{G}, C)\|^2_{\mathcal{H}^2_{\mathrm{HS}}} &= \frac{1}{\tau}\sum_{i=0}^{\infty}\|T_i\|^2_{\mathrm{HS}} = \frac{1}{\tau}\left[\|\hat{D}_{11}\|^2_{\mathrm{HS}} + \sum_{i=1}^{\infty}\operatorname{trace}(T_i{}^* T_i)\right] \\
&= \frac{1}{\tau}\left[\|\hat{D}_{11}\|^2_{\mathrm{HS}} + \sum_{i=1}^{\infty}\operatorname{trace}\left(\hat{B}_1^*(T_{00})_i'\begin{bmatrix}\hat{C}_1^* \\ \tilde{D}_{12}^*\end{bmatrix}\begin{bmatrix}\hat{C}_1 & \tilde{D}_{12}\end{bmatrix}(T_{00})_i\hat{B}_1\right)\right].
\end{aligned}$$

It is a fact that for any two operators A and B such that $\operatorname{trace}(AB) < \infty$ and $\operatorname{trace}(BA) < \infty$, we have

$$\operatorname{trace}(AB) = \operatorname{trace}(BA),$$

since given $\lambda \neq 0$, λ is an eigenvalue of AB if and only if it is an eigenvalue of BA. Using this fact,

$$\begin{aligned}
\|\mathcal{F}(\tilde{G}, C)\|^2_{\mathcal{H}^2_{\mathrm{HS}}} &= \frac{1}{\tau}\left[\|\hat{D}_{11}\|^2_{\mathrm{HS}} + \sum_{i=1}^{\infty} + \sum_{i=1}^{\infty}\operatorname{trace}\left((T_{00})_i'\begin{bmatrix}\hat{C}_1^* \\ \tilde{D}_{12}^*\end{bmatrix}\begin{bmatrix}\hat{C}_1 & \tilde{D}_{12}\end{bmatrix}(T_{00})_i\hat{B}_1\hat{B}_1^*\right)\right] \\
&= \frac{1}{\tau}\left[\|\hat{D}_{11}\|^2_{\mathrm{HS}} + \sum_{i=1}^{\infty}\operatorname{tr}\left((T_{00})_i'\begin{bmatrix}\overline{C}_1' \\ \overline{D}_{12}'\end{bmatrix}\begin{bmatrix}\overline{C}_1 & \overline{D}_{12}\end{bmatrix}(T_{00})_i\overline{B}_1\overline{B}_1'\right)\right] \\
&= \frac{1}{\tau}\left[\|\hat{D}_{11}\|^2_{\mathrm{HS}} + \sum_{i=1}^{\infty}\operatorname{tr}\left(\overline{B}_1'(T_{00})_i'\begin{bmatrix}\overline{C}_1' \\ \overline{D}_{12}'\end{bmatrix}\begin{bmatrix}\overline{C}_1 & \overline{D}_{12}\end{bmatrix}(T_{00})_i\overline{B}_1\right)\right] \\
&= \frac{1}{\tau}\left[\|\hat{D}_{11}\|^2_{\mathrm{HS}} + \|\mathcal{F}(\overline{G}, C)\|^2_{\mathcal{H}^2}\right].
\end{aligned}$$

The last step is arrived at because $\overline{G}$ was constructed so that

$$\overline{G} = \begin{bmatrix}\begin{bmatrix}\overline{C}_1 & \overline{D}_{12}\end{bmatrix} & 0 \\ 0 & I\end{bmatrix}\tilde{G}_{00}\begin{bmatrix}\overline{B}_1 & 0 \\ 0 & I\end{bmatrix},$$

which means that

$$\mathcal{F}(\overline{G}, C) = \begin{bmatrix}\overline{C}_1 & \overline{D}_{12}\end{bmatrix}\mathcal{F}(\tilde{G}_{00}, C)\overline{B}_1 = \begin{bmatrix}\overline{C}_1 & \overline{D}_{12}\end{bmatrix}T_{00}\overline{B}_1. \qquad \square$$

This theorem provides a method of optimizing the $\mathcal{H}^2_{\mathrm{HS}}$ norm of a sampled-data system. From (18) we immediately conclude that a controller C yields an optimum closed-loop $\mathcal{H}^2_{\mathrm{HS}}$ norm, if and only if it (the same controller) is the optimum $\mathcal{H}^2$ controller for the finite dimensional generalized plant $\overline{G}$. To compute the $\mathcal{H}^2_{\mathrm{HS}}$ norm or to perform suboptimal designs, one needs to compute $\|\hat{D}_{11}\|_{\mathrm{HS}}$. Then we simply have

$$\|\mathcal{F}(\tilde{G}, C)\|^2_{\mathcal{H}^2_{\mathrm{HS}}} \leqslant \gamma^2 \iff \|\mathcal{F}(\overline{G}, C)\|^2_{\mathcal{H}^2} \leqslant \gamma^2\tau - \|\hat{D}_{11}\|^2_{\mathrm{HS}}.$$

Thus the $\mathcal{H}^2$ problem for the sampled-data system of Figure 2(b) is equivalent to a standard $\mathcal{H}^2$ problem for the discrete-time generalized plant $\overline{G}$. To actually compute a state space realization of $\overline{G}$, it is not necessary to go through the lifting step: the formulae (16) give the matrices $\hat{A}$, $\hat{C}_2$, $\hat{B}_2$ directly in terms of the matrices of a realization of the original continuous-time plant G. To obtain similar formulae for the remaining matrices $\overline{B}_1$, $\overline{C}_1$, $\overline{D}_{12}$, it is necessary to evaluate the operator compositions on the right hand side of (17). Such compositions involve integration of functions with matrix exponentials, and can be evaluated using the formulae in [8].

To conclude, we briefly summarize these formulae. First, the composition $\hat{B}_1\hat{B}_1^*$ can be evaluated using the matrix exponential

$$\Gamma(\tau)=\begin{bmatrix}\Gamma_{11}(\tau) & \Gamma_{12}(\tau) & \Gamma_{13}(\tau)\\ 0 & \Gamma_{22}(\tau) & \Gamma_{23}(\tau)\\ 0 & 0 & \Gamma_{33}(\tau)\end{bmatrix} := \exp\left\{\tau\begin{bmatrix}-A & I & 0\\ 0 & -A & B_1B_1'\\ 0 & 0 & A'\end{bmatrix}\right\}$$

(the two matrices are partitioned conformably). From [8] it follows that

$$\hat{B}_1\hat{B}_1^*=\int_0^\tau \mathrm{e}^{As}B_1B_1'\,\mathrm{e}^{A's}\,\mathrm{d}s=\Gamma_{33}'(\tau)\Gamma_{23}(\tau).$$

$\Gamma(\tau)$ can also be used to evaluate $\|\hat{D}_{11}\|_{\mathrm{HS}}^2$. From the kernel representation of $\hat{D}_{11}$ (16) and the definition of the Hilbert–Schmidt norm (8), we write

$$\|\hat{D}_{11}\|_{\mathrm{HS}}^2=\mathrm{tr}\left(C_1\left(\int_0^\tau\int_0^s \mathrm{e}^{As}B_1B_1'\,\mathrm{e}^{A's}\,\mathrm{d}s\,\mathrm{d}t\right)C_1'\right),$$

where the integral can be evaluated to be

$$\int_0^\tau\int_0^s \mathrm{e}^{As}B_1B_1'\,\mathrm{e}^{A's}\,\mathrm{d}s\,\mathrm{d}t=\Gamma_{33}'(\tau)\Gamma_{13}(\tau).$$

For the other operator composition, using again the formulae from [8] we can write the kernel function of the operator $[\hat{C}_1 \quad \tilde{D}_{12}]$ as

$$\left[\hat{C}_1(t)\quad \tilde{D}_{12}(t)\right]=[C_1\quad 0]\exp\left\{\begin{bmatrix}A & I\\ 0 & 0\end{bmatrix}t\right\}\begin{bmatrix}I & 0\\ 0 & B_2\end{bmatrix}$$

(here it was assumed for simplicity that the matrix $D_{12}=0$). The integration involved in the operator composition is evaluated using

$$\exp\left\{\begin{bmatrix}\begin{bmatrix}-A' & 0\\ -I & 0\end{bmatrix} & \begin{bmatrix}C_1'C_1 & 0\\ 0 & 0\end{bmatrix}\\ 0 & \begin{bmatrix}A & I\\ 0 & 0\end{bmatrix}\end{bmatrix}\tau\right\}=:\begin{bmatrix}\Phi_{11}(\tau) & \Phi_{12}(\tau)\\ 0 & \Phi_{22}(\tau)\end{bmatrix},$$

where from the formulae in [8] we conclude that

$$\begin{bmatrix}\hat{C}_1^*\\ \tilde{D}_{12}^*\end{bmatrix}\begin{bmatrix}\hat{C}_1 & \tilde{D}_{12}\end{bmatrix}=\begin{bmatrix}I & 0\\ 0 & B_2'\end{bmatrix}\Phi_{22}'(\tau)\Phi_{12}(\tau)\begin{bmatrix}I & 0\\ 0 & B_2\end{bmatrix}.$$

Thus all the matrices in the realization of $\overline{G}$ can be computed directly from the realization of the original continuous-time system G by elementary matrix algebra and matrix exponentiation.

Appendix

We show here that as $n\to\infty$, the quantity in (14) converges to $(1/\tau)\Sigma_{i=0}^{\infty}\|\hat{G}_i\|_{\mathrm{HS}}^2$ (for this section, we drop the subscript $\|\cdot\|_{\mathrm{HS}}$ to simplify notation). To this end, first note that the assumption that the limit in (13) (and equivalently (14)) exists, implies that the sequence $\{\|\hat{G}_i\|\}$ is ℓ^2 summable (if it is not ℓ^2 summable, then the right hand side of (14) can be made arbitrarily large). Second, we check

$$\left|\frac{1}{\tau}\sum_{i=0}^{\infty}\|\hat{G}_i\|^2-\frac{1}{n\tau}\|\Pi_{n\tau}(G)\|^2\right|=\frac{1}{\tau}\left[\frac{1}{n}\|\hat{G}_1\|^2+\cdots+\frac{n-1}{n}\|\hat{G}_{n-1}\|^2+\sum_{i=n}^{\infty}\|\hat{G}_i\|^2\right]. \tag{19}$$

To conclude, we briefly summarize these formulae. First, the composition $\hat{B}_1\hat{B}_1^*$ can be evaluated using the matrix exponential

$$\Gamma(\tau)=\begin{bmatrix}\Gamma_{11}(\tau) & \Gamma_{12}(\tau) & \Gamma_{13}(\tau)\\ 0 & \Gamma_{22}(\tau) & \Gamma_{23}(\tau)\\ 0 & 0 & \Gamma_{33}(\tau)\end{bmatrix}:=\exp\left\{\tau\begin{bmatrix}-A & I & 0\\ 0 & -A & B_1B_1'\\ 0 & 0 & A'\end{bmatrix}\right\}$$

(the two matrices are partitioned conformably). From [8] it follows that

$$\hat{B}_1\hat{B}_1^*=\int_0^\tau \mathrm{e}^{As}B_1B_1'\,\mathrm{e}^{A's}\,\mathrm{d}s=\Gamma_{33}'(\tau)\Gamma_{23}(\tau).$$

$\Gamma(\tau)$ can also be used to evaluate $\|\hat{D}_{11}\|_{\mathrm{HS}}^2$. From the kernel representation of $\hat{D}_{11}$ (16) and the definition of the Hilbert–Schmidt norm (8), we write

$$\|\hat{D}_{11}\|_{\mathrm{HS}}^2=\mathrm{tr}\left(C_1\left(\int_0^\tau\int_0^s \mathrm{e}^{As}B_1B_1'\,\mathrm{e}^{A's}\,\mathrm{d}s\,\mathrm{d}t\right)C_1'\right),$$

where the integral can be evaluated to be

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To show that the right hand side converges to zero, note that the ℓ^2 summability of $\{\|\hat{G}_i\|\}$ implies first that for any $\varepsilon > 0$, there exists N_1 such that the tail sum $\sum_{i=N_1}^{\infty} \|\hat{G}_i\|^2 \leqslant \varepsilon$, and second, that there exists N_2 such that $(1/k)\sum_{i=0}^{\infty} \|\hat{G}_i\|^2 \leqslant \varepsilon$, for all $k \geqslant N_2$. Now choose $n = \max\{N_1, N_2\}$, and observe that

$$\left| \frac{1}{\tau} \sum_{i=0}^{\infty} \|\hat{G}_i\|^2 - \frac{1}{n^2\tau} \|\Pi_{n^2\tau}(G)\|^2 \right|$$

$$= \frac{1}{\tau}\left[\frac{1}{n^2}\|\hat{G}_1\|^2 + \cdots + \frac{n}{n^2}\|\hat{G}_n\|^2 + \cdots + \frac{n^2-1}{n^2}\|\hat{G}_{n^2-1}\|^2 + \sum_{i=n^2}^{\infty} \|\hat{G}_i\|^2 \right]$$

$$\leqslant \frac{1}{\tau}\left[\frac{1}{n}\sum_{i=1}^{n} \|\hat{G}_i\|^2 + \sum_{i=n+1}^{\infty} \|\hat{G}_i\|^2 \right] \leqslant \frac{1}{\tau} 2\varepsilon,$$

and since ε can be arbitrarily small, the right hand side of (19) tends to zero as claimed.

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A lifting technique for linear periodic systems with applications to sampled-data control *

Bassam Bamieh and J. Boyd Pearson

Department of Electrical Engineering, Rice University, Houston, TX 77251, USA

Bruce A. Francis

Department of Electrical Engineering, University of Toronto, Toronto, Canada M5S 1A4

Allen Tannenbaum

Department of Electrical Engineering, University of Minnesota, Minneapolis, MN 55455, USA and Technion, Haifa, Israel

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Abstract: A lifting technique is developed for periodic linear systems and applied to the $\mathscr{H}^\infty$ and $\mathscr{H}^2$ sampled-data control problems.

Keywords: Sampled-data system; lifted system; $\mathscr{H}^\infty$ optimal control; Riccati equation; operator norm.

1. Introduction

Given the success of $\mathscr{H}^\infty$-norm based optimization methods for analog control systems, there has recently been interest in applying such techniques to sampled-data systems [3,4,15,18]. The key point in utilizing such methods would be in their extension to certain periodic time-varying systems. An example of such a system is the sampled-data control system shown in Figure 1 below.

The generalized plant G is a continuous-time, time-invariant system, K_d is discrete-time, time-invariant, S is the ideal periodic sampler with period h, and H the synchronized zero-order hold. Continuous-time signals are represented by continuous lines, discrete-time signals by dotted lines. The behavior of the system from the exogenous input w to the controlled output z is in general time-varying, in fact, periodic with period h.

To analyze the behavior of continuous-time periodic systems, we use a lifting technique similar to that used for discrete-time periodic systems in [16]. Once we develop the lifting technique, we apply it to describe a complete solution to the analysis problem of verifying that a given controller constrains the $\mathscr{L}^2$-induced norm of the sampled-data system to be less than some prespecified level. We will also show that the lifting technique is applicable in fact to all norm-based optimization problems, and in particular to sampled-data versions of the quadratic regulator and optimal filtering problems.

Given the success of $\mathscr{H}^\infty$-norm based optimization methods for analog control systems, there has recently been interest in applying such techniques to sampled-data systems [3,4,15,18,25].

The purpose of this note is to introduce the lifting technique itself and sketch how it can be applied to two optimal control problems. To our knowledge, such a lifting procedure was introduced into sampled-data systems by Toivonen [25], who also treats the $\mathscr{H}^\infty$ sampled-data problem. The details of the lifting in [25] are different from those given here ([25] represents certain finite-rank operators via SVD, which is avoided in our work). The mathematical basis of such lifting techniques may be found in [21]. Reference [1] gives a detailed account of the application of the lifting technique to the $\mathscr{H}^\infty$ sampled-data problem. Yamamoto [28] also uses lifting for sampled-data

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systems, but he lifts the state as well as the input and output. Consequently, his state space is infinite-dimensional, whereas ours is the original finite-dimensional one. Also, Yamamoto treats asymptotic tracking problems, while optimization problems are studied here.

While this paper was being reviewed, several others came into existence. For completeness we mention them here: a sampled-data $\mathscr{H}_2$ problem (different from that in [4]) in [2,17]; sample-data $\mathscr{L}_1$ (i.e., $\mathscr{L}_\infty$ induced norm) in [23,8]; and robust stability of sampled-data systems in [24].

In the operator norm design framework, this lifting technique was developed independently by the first two and the latter two authors. Reference [1] gives a detailed account of the application of this technique to the $\mathscr{H}^\infty$ sampled-data problem.

2. Lifting continuous-time signals

In this section we introduce a construction whereby one may 'lift' a continuous-time signal to a discrete-time one. This construction will also be used to associate a time-invariant discrete-time system to a continuous-time periodic one. The utility of this technique in feedback control is that all norms are preserved, as well as the feedback interconnection structure.

We will first work in a rather general framework before specializing to the case of interest. Let $\mathscr{X}$ denote a Banach space equipped with norm $\|\ \|_{\mathscr{X}}$. For every integer $p \geq 1$ we set

$$\mathscr{L}^p(\mathscr{X}) := \left\{ u : [0, \infty) \to \mathscr{X} : \int_0^\infty \| u(t) \|_{\mathscr{X}}^p \, \mathrm{d}t < \infty \right\}.$$

As is well-known, $\mathscr{L}^p(\mathscr{X})$ is a Banach space with norm

$$\| u \|_{p, \mathscr{X}} := \left(\int_0^\infty \| u(t) \|_{\mathscr{X}}^p \, \mathrm{d}t \right)^{1/p}.$$

For $p = 2$, $\mathscr{L}^2(\mathscr{X})$ may be given the structure of a Hilbert space in the usual way. For $p = \infty$, we have

$$\mathscr{L}^\infty(\mathscr{X}) := \{ u : [0, \infty) \to \mathscr{X} : \operatorname{ess\,sup} \| u(t) \|_{\mathscr{X}} < \infty \}$$

(see, e.g., [22]). Finally, to each of the spaces $\mathscr{L}^p(\mathscr{X})$ we may associate the extended space $\mathscr{L}_e^p(\mathscr{X})$ in the standard way. For all the definitions see [6].

The same types of definitions are of course valid in the discrete-time case for sequences. A sequence will be written as a column vector, for example,

$$\psi = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \end{bmatrix}.$$

Again for any Banach space $\mathscr{X}$, define

$$l^p(\mathscr{X}) = \left\{ \psi : \psi_i \in \mathscr{X}, \sum_{i=0}^\infty \| \psi_i \|_{\mathscr{X}}^p < \infty \right\}, \quad 1 \leq p < \infty,$$

$$l^\infty(\mathscr{X}) = \left\{ \psi : \sup_i \| \psi_i \|_{\mathscr{X}} < \infty \right\}.$$

The norms are given by

$$\| \psi \|_{l^p(\mathscr{X})} = \left(\sum_{i=0}^\infty \| \psi_i \|_{\mathscr{X}}^p \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\| \psi \|_{l^\infty(\mathscr{X})} = \sup_i \| \psi_i \|_{\mathscr{X}}.$$

Equipped with this norm $l^p(\mathscr{X})$ is a Banach space for all $1 \leq p \leq \infty$. Once again for $p = 2$, $l^2(\mathscr{X})$ may be given a Hilbert structure in the usual way [22], and the associated extended space $l_e^p(\mathscr{X})$ may be defined: it is just the linear space of all sequences in $\mathscr{X}$.

We are now ready to describe the lifting procedure. For fixed $h > 0$ let

$$\mathscr{K}^p := \{ u \in \mathscr{L}^p(\mathscr{X}) \text{ with support in } [0, h) \}.$$

Once again $\mathscr{K}^p$ is a Banach space in the natural way with norm induced by $\|\ \|_{p, \mathscr{X}}$. Suppose u is an element of $\mathscr{L}_e^p(\mathscr{X})$. Chop u up into its components as follows:

$$u_0(t) = u(t), \quad 0 \leq t < h,$$
$$u_1(t) = u(t+h), \quad 0 \leq t < h,$$
$$u_2(t) = u(t+2h), \quad 0 \leq t < h,$$

etc.

Each piece, u_i, belongs to $\mathscr{K}^p$. Now form the sequence

$$\psi = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \end{bmatrix}.$$

Define the *lifting operator* W_p to be the map $u \mapsto \psi$. It maps $\mathscr{L}_e^p(\mathscr{X})$ to $l_e^p(\mathscr{K}^p)$. We sometimes write just W when p is irrelevant.

It is important to note that W is a linear bijection from $\mathscr{L}_e^p(\mathscr{X})$ to $l_e^p(\mathscr{K}^p)$ whose inverse is given by

$$u = W^{-1}\psi \quad \Leftrightarrow \quad u(t) = \psi_i(t - hi), \quad hi \le t < h(i+1).$$

It is easy to show that the restriction of W to the Banach space $\mathscr{L}^p(\mathscr{X}) \subset \mathscr{L}_e^p(\mathscr{X})$ is an isometry, $\mathscr{L}^p(\mathscr{X}) \to l^p(\mathscr{K}^p)$.

To recap, W is a bijective linear mapping from $\mathscr{L}_e^p(\mathscr{X})$ to $l_e^p(\mathscr{K}^p)$, and a bijective linear *isometry* from $\mathscr{L}^p(\mathscr{X})$ to $l^p(\mathscr{K}^p)$.

Of course, one may also lift systems. Let $G: \mathscr{L}_e^p(\mathscr{X}) \to \mathscr{L}_e^q(\mathscr{X})$ be a linear operator. Then the lifted system is defined to be $\tilde{G} = W_q G W_p^{-1}$, mapping $l_e^p(\mathscr{K}^p)$ to $l_e^q(\mathscr{K}^q)$. By the linearity of each of the defining operators, $\tilde{G}$ is linear. Moreover, if G is also bounded $\mathscr{L}^p(\mathscr{X}) \to \mathscr{L}^q(\mathscr{X})$, then $\tilde{G}$ is bounded too. Since W_p and W_q are isometries, one sees that $\| G \| = \| \tilde{G} \|$, that is, the system (operator) norm is preserved by the lifting. Furthermore, since the lifting procedure is isometric and preserves all the standard algebraic and feedback interconnection operations, feedback stability is also preserved under lifting.

Now if the system to be lifted is h-periodic, then the lifted system will be time-invariant. To see this, introduce the *delay operator* D_h, defined by $(D_h f)(t) = f(t-h)$. Gives a (causal) system $G: \mathscr{L}_e^p(\mathscr{X}) \to \mathscr{L}_e^q(\mathscr{X})$, we say that G is *h-periodic* if it commutes with D_h, that is, $D_h G = G D_h$. (G is *time-invariant* if it is h-periodic for every $h > 0$.) Let U be the unilateral shift operator on sequences:

$$U \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \psi_0 \\ \psi_1 \\ \vdots \end{bmatrix}, \quad \psi_i \in \mathscr{X}.$$

It is easy to compute that $W_p D_h W_p^{-1} = U$ on $l_e^p(\mathscr{K}^p)$ for any $p \ge 1$. Consequently, for $G: \mathscr{L}_e^p(\mathscr{X}) \to \mathscr{L}_e^q(\mathscr{X})$ h-periodic,

$$\begin{aligned} U\tilde{G} &= W_q D_h W_q^{-1} W_q G W_p^{-1} \\ &= W_q D_h G W_p^{-1} \\ &= W_q G D_h W_p^{-1} \\ &= W_q G W_p^{-1} W_p D_h W_p^{-1} \\ &= \tilde{G} U, \end{aligned}$$

so $\tilde{G}$ is time-invariant. Consequently, $\tilde{G}$ has a convolution representation.

Finally, we remark that all the standard results about the discrete Fourier transform go over to the space $l^2(\mathscr{X})$. We refer the reader to [21] for the details. This may be summarized by the following result.

Proposition 1. (i) *The discrete Fourier transform is an isometric isomorphism from the time-domain space $l^2(\mathscr{X})$ to the frequency-domain space $\mathscr{H}^2(\mathscr{X})$ (the space of square integrable $\mathscr{X}$-valued analytic functions defined on the unit disk).*

(ii) *If G is a bounded analytic $\mathscr{X}$-valued function on the unit disk, it defines a bounded operator on $\mathscr{H}^2(\mathscr{X})$ by multiplication, and its induced norm equals exactly $\| G \|_\infty$.*

By the equivalence between an h-periodic system and its lifting, this theorem provides a 'frequency-domain' characterization of the $\mathscr{L}^2$-induced norm of an h-periodic system.

3. Lifting: some examples

Now we look at what lifting means for state-space models. In what follows, G is a continuous-time finite-dimensional time-invariant linear system. Its input, state, and output evolve in finite-dimensional Euclidean spaces. Because the dimensions of these spaces will be irrelevant, they will all be denoted by $\mathscr{E}$. Thus G is considered as a linear operator on $\mathscr{L}_e^2(\mathscr{E})$. Suppose it has the realization A, B, C, D.

3.1. Lifting G

We begin by lifting G itself. The lifted system, WGW^{-1}, acts on $l_e^2(\mathscr{K}^2)$ and consequently has a matrix representation of the form

$$\begin{bmatrix} G_{11} & 0 & 0 & \cdots \\ G_{21} & G_{22} & 0 & \cdots \\ G_{31} & G_{32} & G_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

It is important to note that W is a linear bijection from $\mathscr{L}_e^p(\mathscr{X})$ to $l_e^p(\mathscr{K}^p)$ whose inverse is given by

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Starting with $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ as above, define the operators

$$Q = I - \tilde{D}\tilde{D}^*, \quad R = I - \tilde{D}^*\tilde{D}$$

mapping $\mathscr{K}^2$ to $\mathscr{K}^2$, and define the pencil

$$S = \lambda \begin{bmatrix} I & -\tilde{B}R^{-1}\tilde{B}^* \\ 0 & \tilde{A}^* + \tilde{C}^*\tilde{D}R^{-1}\tilde{B}^* \end{bmatrix} - \begin{bmatrix} \tilde{A} + \tilde{B}R^{-1}\tilde{D}^*\tilde{C} & 0 \\ -\tilde{C}^*Q^{-1}\tilde{C} & I \end{bmatrix}.$$

Observe, for example, that $\tilde{B}R^{-1}\tilde{B}^*$ maps $\mathscr{E}$ to $\mathscr{E}$, i.e., it is a finite matrix. So S is a finite matrix pencil. Suppose S has no eigenvalues on the unit circle; then it must have n inside the unit disc. Let $\mathscr{X}_-$ denote the corresponding spectral subspace. It can be represented as

$$\mathscr{X}_- = \operatorname{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

where X_1 and X_2 are both $n \times n$. Assuming X_1 is invertible, we can define $X := X_2 X_1^{-1}$. This defines the Riccati operator Ric: $S \mapsto X$ and its domain. Lemma 2.3 of [14] provides the following.

Lemma 1. $\| WGW^{-1} \| < 1$ *iff the following three conditions hold*:
(a) $\| \tilde{D} \| < 1$;
(b) *S belongs to the domain of* Ric;
(c) $R - \tilde{B}^* X B > 0$, *where* $X = \operatorname{Ric}(S)$.

To compute $\| WGW^{-1} \|$ in this way, we would have to
- compute the matrices $\tilde{B}R^{-1}\tilde{B}^*$, $\tilde{C}^*Q^{-1}\tilde{C}$, $\tilde{B}R^{-1}\tilde{D}^*\tilde{C}$ in the definition of S,
- compute $\| \tilde{D} \|$, and
- check if $R - \tilde{B}^* X B > 0$ for a given matrix X.

These subproblems are similar. We will mention two methods for the second subproblem. First of all, let $\Pi: \mathscr{L}^2(\mathscr{E}) \to \mathscr{K}^2$ denote orthogonal projection. Observe that $\tilde{D}$ is the compression of the unlifted system G to $\mathscr{K}^2$, i.e., $\tilde{D} = \Pi G | \mathscr{K}^2$. Note that the Laplace transform is an isomorphism of $\mathscr{K}^2$ onto $\mathscr{H}^2 \ominus e^{-hs}\mathscr{H}^2$. Thus computing $\| \tilde{D} \|$ amounts to computing the norm of the operator 'multiplication by the transfer matrix for G' compressed to $\mathscr{H}^2 \ominus e^{-hs}\mathscr{H}^2$. In [9] this computation is reduced to a linear two-point boundary value problem. See also [1]. In [10] a second, frequency-domain ('skew Toeplitz') approach is given for the computation of $\| \tilde{D} \|$.

In summary, the computation of $\| WGW^{-1} \|$ involves the standard iteractive search of scaling, and then using Lemma 1 to check if the norm of the scaled system is less than one.

3.2. Lifting SG

The ideal sampling operator with period h is defined by

$$\psi = Su \;\Rightarrow\; \psi(k) = u(kh).$$

We shall lift SG, where G is as before except with $D = 0$. Operator SG maps $\mathscr{L}_e^2(\mathscr{E})$ to $l_e^2(\mathscr{E})$; G is assumed strictly causal so that SG is bounded on these spaces. The output from SG is already discrete-time, so we need lift only the input. The lifted system, SGW^{-1}, acts from $l_e^2(\mathscr{K}^2)$ to $l_e^2(\mathscr{E})$. Its matrix is easily derived to be

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right]$$

where

$$\tilde{A}: \mathscr{E} \to \mathscr{E}, \quad \tilde{A}x = e^{hA}x,$$

$$\tilde{B}: \mathscr{K}^2 \to \mathscr{E}, \quad \tilde{B}u = \int_0^h e^{(h-\tau)A} Bu(\tau)\, d\tau,$$

$$\tilde{C}: \mathscr{E} \to \mathscr{E}, \quad \tilde{C}x = Cx.$$

3.3. Lifting GH

Finally, we shall lift GH, where H is the ideal hold operator with period h, defined by

$$y = H\psi \;\Leftrightarrow\; y(t) = \psi(k), \quad kh \le t < (k+1)h.$$

This is an operator from $l_e^2(\mathscr{E})$ to $\mathscr{L}_e^2(\mathscr{E})$. The input to GH is already discrete-time, so we need lift only the output. The lifted system, WGH, acts from $l_e^2(\mathscr{E})$ to $l_e^2(\mathscr{K}^2)$ and its matrix is

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$$

where

$$\tilde{A}: \mathscr{E} \to \mathscr{E}, \quad \tilde{A}x = e^{hA}x,$$

$$\tilde{B}: \mathscr{E} \to \mathscr{E}, \quad \tilde{B}v = \int_0^h e^{\tau A}\, d\tau B v,$$

$$\tilde{C}: \mathscr{E} \to \mathscr{K}^2, \quad (\tilde{C}x)(t) = C e^{tA}x,$$

$$\tilde{D}: \mathscr{E} \to \mathscr{K}^2, \quad (\tilde{D}v)(t) = \left[D + \int_0^t C e^{\tau A}\, d\tau B \right] v.$$

4. Application to $\mathcal{H}^\infty$ optimization of sampled-data systems

In this section we outline an application to optimizing the $\mathcal{L}^2(\mathcal{E})$-induced norm from w to z in Figure 1. Let T denote the linear system mapping w to z. If K_d is internally stabilizing (suitably defined) and under mild assumptions on G, T is a bounded operator on $\mathcal{L}^2(\mathcal{E})$. It is time-varying. Our approach is to lift T up to WTW^{-1}, which will be a time-invariant operator on $l^2(\mathcal{K}^2)$. The optimization of $\|T\|$ is thus reduced to a discrete-time, time-invariant $\mathcal{H}^\infty$ optimization problem, a problem whose solution is formally the same as the standard discrete-time matrix-valued $\mathcal{H}^\infty$ problem for which there exist solutions [13,14,19,20]. (An alternative but equivalent approach is taken in [1] where the operator valued $\mathcal{H}^\infty$ problem is solved through an intermediate step of reducing it to an equivalent matrix-valued discrete-time $\mathcal{H}^\infty$ problem.)

The details of our approach are as follows. Partition G as

$$G=\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

and let a corresponding minimal realization be

$$\left[\begin{array}{c|c} A & \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\ \hline \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix} \end{array}\right].$$

In Figure 1, bring S and H around and adsorb them into G to get the setup shown in Figure 2 below. Matrix D_{21} is taken to be zero so that w is low-pass filtered (through G_{21}) before being sampled; the system could not in general be internally stabilized without this assumption.

The system in the upper block is

$$\begin{bmatrix} G_{11} & G_{12}H \\ SG_{21} & SG_{22}H \end{bmatrix}.$$

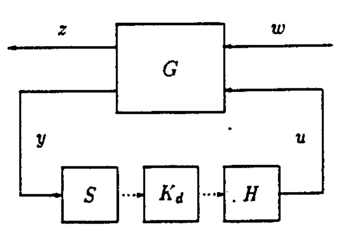

Fig. 1. Sampled-data control system.

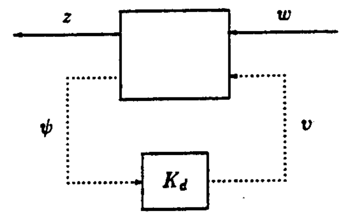

Fig. 2.

Now lift w and z in the previous figure to arrive at the setup in Figure 3 below.

System P is obviously given by

$$P=\begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} G_{11} & G_{12}H \\ SG_{21} & SG_{22} \end{bmatrix}\begin{bmatrix} W^{-1} & 0 \\ 0 & I \end{bmatrix}$$

$$=\begin{bmatrix} WG_{11}W^{-1} & WG_{12}H \\ SG_{21}W^{-1} & SG_{22}H \end{bmatrix}.$$

Realizations of the three liftings $WG_{11}W^{-1}$, $WG_{12}H$, $SG_{21}W^{-1}$ were obtained in Section 3. Furthermore, $SG_{22}H$ is just G_{22} discretized: a realization is well-known to consist of the four matrices

$$e^{hA}, \quad \int_0^h e^{\tau A}\,d\tau B_2, \quad C_2, \quad D_{22}.$$

In this way we get the realization of P,

$$\left[\begin{array}{c|c} \tilde{A} & \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 \end{bmatrix} \\ \hline \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & \tilde{D}_{22} \end{bmatrix} \end{array}\right]$$

where

$$\tilde{A}:\mathcal{E}\to\mathcal{E}, \quad \tilde{A}x=e^{hA}x,$$

$$\tilde{B}_1:\mathcal{K}^2\to\mathcal{E}, \quad \tilde{B}_1w=\int_0^h e^{(h-\tau)A}B_1w(\tau)\,d\tau,$$

$$\tilde{B}_2:\mathcal{E}\to\mathcal{E}, \quad \tilde{B}_2v=\int_0^h e^{\tau A}\,d\tau B_2v,$$

$$\tilde{C}_1:\mathcal{E}\to\mathcal{K}^2, \quad (\tilde{C}_1x)(t)=C_1\,e^{tA}x,$$

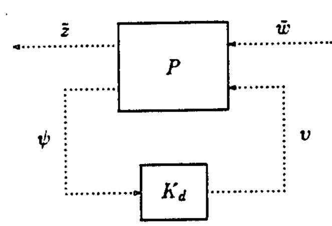

Fig. 3.

$$\tilde{C}_2 : \mathscr{E} \to \mathscr{E}, \quad \tilde{C}_2 x = C_2 x,$$

$$\tilde{D}_{11} : \mathscr{K}^2 \to \mathscr{K}^2,$$

$$(\tilde{D}_{11} w)(t) = D_{11} w(t) + C_1 \int_0^t e^{(t-\tau)A} B_1 w(\tau) \, d\tau,$$

$$\tilde{D}_{12} : \mathscr{E} \to \mathscr{K}^2,$$

$$(\tilde{D}_{12} v)(t) = D_{12} v + C_1 \int_0^t e^{(t-\tau)A} \, d\tau B_2 v,$$

$$\tilde{D}_{22} : \mathscr{E} \to \mathscr{E}, \quad \tilde{D}_{22} v = D_{22} v.$$

Figure 3 is a discrete-time setup. Iglesias and Glover's solution [14] to the discrete-time $\mathscr{H}^\infty$ problem is in the style of the continuous-time solution of Doyle et al. [7]. We will illustrate how the solution of [14] can be applied to the setup at hand by looking at the analysis problem, which is easier than the synthesis problem. Namely, for a fixed stabilizing K_d we will show how to compute the $\mathscr{L}^2(\mathscr{E})$-induced norm.

In Figure 3 the equations for P are

$$\xi_P(k+1) = \tilde{A}\xi_P(k) + \tilde{B}_1 w_k + \tilde{B}_2 v(k),$$

$$z_k = \tilde{C}_1 \xi_P(k) + \tilde{D}_{11} w_k + \tilde{D}_{12} v(k),$$

$$\psi(k) = \tilde{C}_2 \xi_P(k) + \tilde{D}_{22} v(k).$$

Suppose K_d is strictly causal for simplicity, and its equations are

$$\xi_K(k+1) = A_K \xi_K(k) + B_K \psi(k),$$

$$v(k) = C_K \xi_K(k).$$

Then the matrix of the closed-loop system is

$$\left[\begin{array}{c|c} A_{CL} & B_{CL} \\ \hline C_{CL} & D_{CL} \end{array}\right]$$

where A_{CL} is the map from $\mathscr{E}$ to $\mathscr{E}$ given by

$$A_{CL}\xi = \begin{bmatrix} \tilde{A} & \tilde{B}_2 C_K \\ B_K \tilde{C}_2 & A_K \end{bmatrix} \xi = \begin{bmatrix} e^{hA} & \int_0^h e^{\tau A} \, d\tau B_2 C_K \\ B_K C_2 & A_K \end{bmatrix} \xi,$$

B_{CL} is the map from $\mathscr{K}^2$ to $\mathscr{E}$ given by

$$B_{CL} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix},$$

C_{CL} is the map from $\mathscr{E}$ to $\mathscr{K}^2$ given by

$$C_{CL} = \begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} C_K \end{bmatrix},$$

and $D_{CL} = \tilde{D}_{11}$, mapping $\mathscr{K}^2$ to $\mathscr{K}^2$. Internal stability means that all eigenvalues of A_{CL} are inside the unit disk. Computing now proceeds as in Subsection 3.1.

5. Application to $\mathscr{H}^2$ optimization of sampled-data systems

In this section, we would like to make some remarks about the lifting technique applied to other types of norms. Since the lifting is an isometry in any given norm, we can apply it to other $\mathscr{L}^p$ spaces. First we would like to make some remarks about the induced operator norm on l^p.

Consider $\mathscr{E}$ equipped with the l^r-norm,

$$\| v \|_r = \left[\sum_{j=1}^{n} | v_j |^r \right]^{1/r}, \quad 1 \le r < \infty,$$

$$\| v \|_\infty = \max_{1 \le j \le n} | v_j |,$$

where the v_j denote the components of the vector $v \in \mathscr{E}$. With $\mathscr{E}$ equipped with the r-norm we will set $\mathscr{L}_r^p(\mathscr{E}) := \mathscr{L}^p(\mathscr{E})$ and denote the norm by $\| \ \|_{p,r}$. Also, $\mathscr{K}_r^p$ will denote the subspace of $\mathscr{L}_r^p(\mathscr{E})$ of functions with support in $[0, h)$. By slight abuse of notation, $\| \ \|_{p,r}$ will also denote the norm on $\mathscr{K}_r^p$.

By the lifting construction, we see that there exists an isometry $W_{p,r} : \mathscr{L}_r^p(\mathscr{E}) \to l^p(\mathscr{K}_r^p)$ for each $1 \le p, r \le \infty$. Recall that the induced norm of a bounded linear operator T from one Banach space $\mathscr{X}_1$ to another Banach space $\mathscr{X}_2$ is

$$\| T \| := \sup_{v \neq 0} \frac{\| Tv \|_{\mathscr{X}_2}}{\| v \|_{\mathscr{X}_1}}.$$

We consider the problem, then, of computing the induced norm of a discrete-time causal convolution operator $F : l^p(\mathscr{K}_r^p) \to l^q(\mathscr{K}_s^q)$. When $p = q = r = s = 2$ we have seen that the induced norm is in fact the $\mathscr{H}^\infty$-norm of the discrete Fourier transform of the pulse response of F. But this of course is not the only possibility, and one can ask for choices of p, q, r, s which will induce a 2-norm which would correspond to a quadratic type sampled-data optimization problem. We should note that in [4] the authors consider an optimization problem with the Hilbert–Schmidt norm, which is not an operator-induced norm.

Before stating the result, we will need some additional notation. First if $F: l^p(\mathscr{K}_r^p) \to l^q(\mathscr{K}_s^q)$ is a (causal) convolution operator, the equality $y = F(u)$ means that

$$y_k = \sum_{i=0}^{k} F_{k-i}(u_i), \quad \forall k \geq 0,$$

where the $F_k: \mathscr{K}_r^p \to \mathscr{K}_s^q$ are linear operators. In our case, the F_k will come from the lifted closed-loop operator T_{zw}, and so will have the form

$$C_{\mathrm{CL}} A_{\mathrm{CL}}^{k-1} B_{\mathrm{CL}}, \quad k \geq 1, \tag{2}$$

in the notation of Section 4. Note that the closed-loop impulse response F_k for $k = 0$ is the operator $\tilde{D}_{11}$. This fact will not affect our discussion below, since the controller only enters into the closed loop operator F_k for $k \geq 1$, and so from the way in which the norm is computed, the controller that minimizes the cost (norm) with F_0 included is the same as the controller that minimizes the cost without including F_0, hence the answer given below is valid.

Next notice that

$$B_{\mathrm{CL}} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix},$$

where

$$\tilde{B}_1 w = \int_0^h \mathrm{e}^{(h-\tau)A} B_1 w(\tau)\, \mathrm{d}\tau.$$

Thus $\tilde{B}_1$ acts as a convolution operator evaluated at h, and so we may express the action of the impulse response function (2) as an integral operator of the form

$$(F_k u)(t) = \int_0^h F_k(h-\tau, t) u(\tau)\, \mathrm{d}\tau, \quad k \geq 1.$$

Now for A a non-negative matrix, we let $\lambda_{\max}(A)$ denote the maximal eigenvalue, and $d_{\max}(A)$ the maximal diagonal entry.

The following result may be proven using a method similar to that in [26].

Proposition 2. *For all* $k \geq 1$, *set*

$$Q_1^k(\tau) := \int_0^h F_k(h-\tau, t)' F_k(h-\tau, t)\, \mathrm{d}t,$$

$$Q_2^k(\tau) := \int_0^h F_k(h-\tau, t) F_k(h-\tau, t)'\, \mathrm{d}t,$$

for $\tau \in [0, h)$, *and set*

$$R_1^k := \sup_{\tau \in [0,h)} \lambda_{\max}\big(Q_1^k(\tau)\big),$$

$$R_2^k := \sup_{\tau \in [0,h)} \lambda_{\max}\big(Q_2^k(\tau)\big),$$

$$S_1^k := \sup_{\tau \in [0,h)} d_{\max}\big(Q_1^k(\tau)\big),$$

$$S_2^k := \sup_{\tau \in [0,h)} d_{\max}\big(Q_2^k(\tau)\big).$$

Then (i) *the induced norm of* $F: l^1(\mathscr{K}_2^1) \to l^2(\mathscr{K}_2^2)$ *equals*

$$\Big(\sum_k R_1^k\Big)^{1/2};$$

(ii) *the induced norm of* $F: l^2(\mathscr{K}_2^2) \to l^\infty(\mathscr{K}_2^\infty)$ *equals*

$$\Big(\sum_k R_2^k\Big)^{1/2};$$

(iii) *the induced norm of* $F: l^1(\mathscr{K}_1^1) \to l^2(\mathscr{K}_2^2)$ *equals*

$$\Big(\sum_k S_1^k\Big)^{1/2};$$

(iv) *the induced norm of* $F: l^2(\mathscr{K}_2^2) \to l^\infty(\mathscr{K}_\infty^\infty)$ *equals*

$$\Big(\sum_k S_2^k\Big)^{1/2}.$$

Referring again to Figure 1, we can pose the problem of minimizing the operator norm of the transfer operator from w to z, where we allow the signals to be in the various spaces $\mathscr{L}_r^p(\mathscr{E})$. This problem may be lifted to get the equivalent discrete-time problem in the spaces $l^p(\mathscr{K}_r^p)$ and then one may apply the solution in [27]. For the full state information problem (this corresponds to the classical LQR problem) one can show that the classical LQR optimal controller is optimal in the case when the disturbances are in $\mathscr{L}_r^1$ for $r = 1, 2$, and the errors are in $\mathscr{L}_2^2$. For the optimal filtering problem, one can show that the optimal state estimator is again given by the classical formula with disturbances in $\mathscr{L}_2^2$ and errors in $\mathscr{L}_r^\infty$ for $r = 2, \infty$. The argument goes exactly as in the $\mathscr{H}^\infty$ case by considering the equivalent 'lifted' discrete

time-invariant system and applying Proposition 2 and the results of [27]. Note that from our previous remarks the lifted operator $\hat{G}$ is finite dimensional.

To make this argument more concrete, we will consider the sampled-data version of the full state information (LQR) problem. Referring to Section 4, in this case the generalized plant G has the form

$$\left[\begin{array}{c|c} A & \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\ \hline \begin{bmatrix} C_1 \\ I \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix} \end{array}\right].$$

We are interested in minimizing the induced operator norm of T, the linear input/output operator from w to z taken over all the controllers K as in Figure 1. For our problem, we assume that $w \in \mathscr{L}_r^1$ ($r = 1$ or $r = 2$) and $z \in \mathscr{L}_2^2$.

Now in this case the lifted system will have the form

$$\left[\begin{array}{c|c} \tilde{A} & \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 \end{bmatrix} \\ \hline \begin{bmatrix} \tilde{C}_1 \\ I \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & 0 \end{bmatrix} \end{array}\right]. \tag{3}$$

Note once again that all norms are preserved in the lifting procedure. Hence, arguing precisely as in Section 4 (and making the standard assumptions of stabilizability and detectability on (3)), and using the results of [27], the optimal feedback gain may be derived from the classical finite dimensional algebraic Stein (discrete Riccati equation) associated to the LQR problem with respect to the generalized time-invariant, discrete-time plant given in (3).

Unfortunately, at this point there is no separation principle available because of the incompatibility of the norms in the filtering and regulator problems.

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KERNEL REPRESENTATION AND PROPERTIES OF DISCRETE-TIME INPUT-OUTPUT SYSTEMS

by

A. P. KISHORE and J. B. PEARSON, JR.

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Kernel Representation and Properties of Discrete-Time Input-Output Systems*

A. P. Kishore and J. B. Pearson, Jr.
Department of Electrical and Computer Engineering
Rice University
Houston, Texas 77251-1892

Submitted by A. C. Antoulas

ABSTRACT

Discrete-time systems in a formal input-output setting are considered. Weak linearity, weak shift invariance, and weak nonanticipation are defined. The often overlooked fact that linear systems may not have a kernel representation is pointed out. Necessary and sufficient conditions for kernel representation on l_p spaces are given. It is shown that a linear system can have infinitely many kernel representations and that properties such as nonanticipation, shift invariance, and boundedness need not be reflected in the structure of a kernel representation. It is argued that a system is logically distinct from a parametric representation of itself.

NOTATION AND TERMINOLOGY

We denote the set of integers by Z, the set of nonnegative integers by Z_+. The sequence space on Z is denoted by $l(Z)$ and called the bilateral sequence space; that on Z_+ is denoted by $l(Z+)$ and called the unilateral sequence space. When a statement is true for both $l(Z)$ and $l(Z_+)$, we write l. We denote the time set associated with l by T. If $l = l(Z)$ $[l(Z_+)]$, then $T = Z\,[Z_+]$. For a fixed n in T, $\delta_n \in l$ denotes the (unit-impulse) sequence which has value 1 at n and 0 everywhere else. For maps, $\mathscr{D}$ denotes the

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domain, $\mathscr{R}$ the range. The (right) shift operator on both $l(Z_+)$ and $l(Z)$ is denoted by the same letter S, and is defined below:

$S : l(Z_+) \mapsto l(Z_+)$ is given by

$$(Sx)(n) = \begin{cases} 0 & \text{if } n = 0, \\ x(n-1) & \text{if } n > 0. \end{cases}$$

$S : l(Z) \mapsto l(Z)$ is given by

$$(Sx)(n) = x(n-1) \qquad \forall n \in Z.$$

Likewise, the symbol S^{-1} denotes the left-shift operator on both $l(Z_+)$ and $l(Z)$ and is defined by

$$(S^{-1}x)(n) = x(n+1) \qquad \forall n \in T.$$

The symbol S_n denotes the selection functional that selects the nth coordinate of a sequence.

The symbol $P_n : l \mapsto l$ denotes the projection operator on two spaces:

$$(P_n x)(i) = \begin{cases} x(i) & \text{if } i \leqslant n, \\ 0 & \text{if } i > n. \end{cases}$$

Finally, if $x \in l$, then x_l^u denotes the sequence defined by

$$x_l^u(i) = \begin{cases} x(i) & \text{if } l \leqslant i \leqslant u, \\ 0 & \text{otherwise.} \end{cases}$$

A subset $X \subset l$ is said to be closed under the family of projections $\{P_n\}$, $n \in T$, if for each $n \in T$, $x \in X$ implies $P_n x \in X$.

I. INTRODUCTION

An input-output system is a relation between two function spaces. The classical input-output framework treats a system as a *map* from one function space into another. Associated with a map are its topological properties such as boundedness and nontopological properties such as shift invariance. The

collection of all input-output pairs associated with the map is called the graph (also, behavior) of the map. The properties of the map are naturally related to its graph. Sometimes the action of a map may admit a concrete representation such as matrix multiplication in the case of sequence spaces, or a Volterra integral representation in the case of spaces of functions of a real variable. Such a representation, if exists, may or may not be unique; it may or may not reflect in its structure the properties of the associated graph. For example, a shift-invariant map on a sequence space may have a representation as an infinite Toeplitz matrix. It is of interest to know when a behavior admits a representation, if a representation is unique, and if a representation reflects the properties of interest.

Maps on sequence spaces are considered here. In this paper, representation means kernel representation, which will be defined in the next section. Representation is the main focus of the paper. We give necessary and sufficient conditions for kernel representation on l_p spaces. We also examine the relationship of kernel representation with properties like shift invariance and nonanticipation. We also show that a representation need not be unique and give a sufficient condition for uniqueness. It is implicit (sometimes explicit) in textbooks on systems theory that a representation always exists and in its structure reflects the properties of the associated behavior. We point out that this is not true. Therefore, it is the behavior that is fundamental, not its representation [6]. We also look at conventional definitions of properties such as shift invariance and point out that they lead to anomalies between maps on bilateral sequence spaces and unilateral sequence spaces. We propose new definitions of properties of maps and argue their merit. The new definitions also make it clear what properties of domains are or are not used in the analysis. However, the new definitions are not the main aspect of the paper. A deep analysis of the differences between maps on bilateral and unilateral sequence spaces is not attempted here.

2. SOME PROPERTIES OF MAPS

The main practical reason for studying linear mathematics is that local behavior of a nonlinear map is often linear. That is, if the domain of a given nonlinear map is restricted, the restricted map (the *restriction*) may become linear, thereby making analysis easier. Then, if the domain is restricted further, it is desirable for the resulting restriction still to be linear. Considering that linearity is an analytically desirable property of a map, all the restrictions of a linear map should inherit this property. Similarly, inheritance by restrictions is desirable with respect to shift invariance, nonanticipation,

and boundedness, from a practical point of view. Consider the classical definition of linearity below:

DEFINITION 2.1. A map $G: \mathscr{D}(G) \to \mathscr{R}(G)$ is *linear* if $\mathscr{D}(G)$ is a linear space and if $G(\alpha x + \beta y) = \alpha Gx + \beta Gy \ \forall \alpha, \beta \in \mathbb{R}, \forall x, y \in \mathscr{D}(G)$.

According to the above definition, the identity map on l_2 is linear but the identity map on the unit ball of l_2 is not. That is, linearity is not necessarily inherited by restrictions. It seems reasonable to call the identity map linear, whether or not its input class is a linear space. We now consider another nontopological property, nonanticipation. There are two definitions in the classical framework for nonanticipation, with one leading to inheritence, and one not. The following definition is in, e.g., [5].

DEFINITION 2.2. A map $G: \mathscr{D}(G) \subseteq l \to \mathscr{R}(G) \subseteq l$ is *nonanticipatory* if $\mathscr{D}(G)$ is closed under the family of projections $\{P_n\}$, $n \in T$, and if $P_n G P_n = P_n G \ \forall n \in T$.

If a map is nonanticipatory according to this definition, its restrictions need not be. The reason is that the domain of a restriction need not be closed under the family of projections. Now consider another notion for nonanticipation, which is in, e.g., [7].[1]

DEFINITION 2.3. The map $G: \mathscr{D}(G) \subseteq l \to \mathscr{R}(G) \subseteq l$ is *weakly nonanticipatory* if for all $n \in T$, $P_n x_1 = P_n x_2$, $x_1, x_2 \in \mathscr{D}(G)$ implies that $P_n G x_1 = P_n G x_2$.

It is evident that if a map is weakly nonanticipatory, all its restrictions also are. Every nonanticipatory map is weakly nonanticipatory. That is, the assumption that a map is weakly nonanticipatory is weaker than the assumption that the map is nonanticipatory. In case the domain of the map is closed under the family of projections, a map is weakly nonanticipatory if and only if it is nonanticipatory.

In the fashion of the weak nonanticipation, we define weak linearity below:

DEFINITION 2.4. A map $G: \mathscr{D}(G) \to \mathscr{R}(G)$ is *weakly linear* if $x, y, \alpha x + \beta y \in \mathscr{D}(G)$, $\alpha, \beta \in \mathbb{R}$ implies that $G(\alpha x + \beta y) = \alpha Gx + \beta Gy$.

[1] In [7], "nonanticipatory" is used instead of "weakly nonanticipatory" in the definition.

Again, every linear map is weakly linear. If $\mathscr{D}(G)$ is a linear space, then a map is weakly linear if and only if it is linear. Also, if a map is weakly linear, so are its restrictions. For example, the identity map is weakly linear whether or not its domain is a linear space. We now define weak shift invariance, a nontopological property of a map.

DEFINITION 2.5. A map $G : \mathscr{D}(G) \subseteq l \to \mathscr{R}(G) \subseteq l$ is *weakly shift-invariant* on $\mathscr{D}(G)$ if for each $x \in \mathscr{D}(G)$ such that $Sx \in \mathscr{D}(G)$, we have $SGx = GSx$.

We say that a subset X of l is shift-invariant if $SX \subset X$. The standard definition of shift invariance follows.

DEFINITION 2.6. A map $G : \mathscr{D}(G) \subseteq l \to \mathscr{R}(G) \subseteq l$ is *shift-invariant* on $\mathscr{D}(G)$ if $\mathscr{D}(G)$ is shift-invariant and if $GS = SG$ on $\mathscr{D}(G)$.

Every shift-invariant map is weakly shift-invariant. If the domain of the map is shift-invariant, then a map is weakly shift-invariant if and only if it is shift-invariant. It is customary to define shift invariance for a system operating on unilateral sequence space $l(Z_+)$ only when the system is nonanticipatory [5]. Nonanticipation is not mentioned in definition of (weak) shift invariance above. The reason for the custom and for our omission will be apparent shortly.

Compared to the standard definitions, the corresponding requirement on the domain of a map is dropped in the new definitions. This does not mean that the domains do not play any role in the properties of a system. On the contrary, the domain is an integral part of a map on which properties of a system do depend. For instance, a map may not be linear but its restrictions may be. Domains play an important role in extension problems, and attention should be paid to what properties continue to hold for the extended map. For instance, a map can be linear and shift-invariant on its domain, and there may be an abvious linear extension of the map to a set containing the domain, but the linear extension may not be shift-invariant. To illustrate this, the following easy proposition is needed.

PROPOSITION 2.7. *Let $G : \mathscr{D}(G) \to \mathscr{R}(G)$ be (weakly) shift-invariant and one-to-one on $D(G)$. If $HG = I$ on $\mathscr{D}(G)$, then H is (weakly) shift-invariant on $\mathscr{R}(G)$.*

A proposition in terms of conventional definitions about maps on unilateral sequence spaces follows.

PROPOSITION 2.8. *Let* $G:\mathscr{D}(G)\subset l(Z_+)\mapsto l(Z_+)$ *be linear and shift-invariant. If* $S^{-1}\mathscr{D}(G)\subset\mathscr{D}(G)$ *then* G *is nonanticipatory.*

The proof is omitted, as it is trivial. However, it should be noted that the hypothesis that $S^{-1}\mathscr{D}(G)\subset\mathscr{D}(G)$ is important for the conclusion: Take $H=S$ with $\mathscr{D}(H)=l(Z_+)$. Consider its inverse $G=S^{-1}$ on $\mathscr{D}(G)=\mathscr{R}(H)=\{x\in l(Z_+): x(0)=0\}$. Clearly, G is linear. That G is shift-invariant follows from Proposition 2.7. But $G=S^{-1}$ is anticipatory on $\mathscr{D}(G)$. Also, while G is linear and shift-invariant on its domain, its obvious linear extension to all of $l(Z_+)$ is not shift-invariant. The above proposition is false if $l(Z_+)$ is replaced by $l(Z)$.

It follows that every linear, shift-invariant $G:\mathscr{D}(G)=l_2(Z_+)\mapsto l_2(Z_+)$ is nonanticipatory on $l_2(Z_+)$. (This appears to be the reason for the custom mentioned above.)

However, every $x\in l(Z_+)$ can be trivially embedded in $l(Z)$ as $\tilde{x}$ below:

$$\tilde{x}(i)=\begin{cases}x(i) & \text{if } i\geqslant 0,\\ 0 & \text{otherwise.}\end{cases}$$

Since the graph of a map on a unilateral sequence space is simply a collection of pairs of unilateral sequences, it is also a collection of pairs of bilateral sequences, by the above canonical embedding. This is the canonical embedding of a system on one-sided sequence spaces into the set of systems on two-sided sequence spaces [5]. Therefore, a given graph on a unilateral sequence space can be analyzed in two ways: by treating its graph as a collection of pairs of unilateral sequences or as a collection of pairs of bilateral sequences. It is remarked in [5] that it is easier to perform certain calculations with the time set Z and then to draw conclusions for Z_+. The point is that the conclusions should be identical with both kinds of analysis. This is not the case with conventional definitions of linearity and shift invariance: Proposition 2.8 is false if G is treated as a map on bilateral sequences with canonical embedding. [With canonical embedding, $\mathscr{D}(G)$ being a linear and shift-invariant space does not imply that it is closed under the family of projections. Example 4 in the next section demonstrates this point.]

Clearly, the conventional definitions lead to an anomaly in drawing conclusions for maps on unilateral sequence spaces and bilateral sequence spaces, depending on whether the graph of a map is treated as a collection of pairs on a unilateral sequence space or as a collection of pairs on a bilateral sequence space.

With respect to this anomaly, the new definitions fare better. We now show that Proposition 2.8 is false with each property replaced by the corresponding weaker property even when the analysis is done without the embedding.

EXAMPLE 1. Let $X = \{x \in l(Z_+): \forall i \in Z_+,\ x(i) \neq 0\}$. Let $G: X \mapsto X$ be given by $(Gx)(n) = x(n+1)\ \forall n \in Z_+$. The domain of G, X, is not a linear space and is not shift-invariant. However, G is weakly linear on X and vacuously weakly shift-invariant on X. Moreover, $S^{-1}X \subset X$. However, G is not weakly nonanticipatory on X.

We now look at the representation aspect of input-output systems.

3. KERNEL REPRESENTATIONS

Suppose the graph of a map on l is given. Let $t_0 := \inf T$. A map $G: \mathscr{D}(G) \subseteq l \to \mathscr{R}(G) \subseteq l$ is said to have a *kernel representation* if there exists a $g: T \times T \to \mathbb{R}$ such that

$$(Gu)(n) = \sum_{m=t_0}^{\infty} g(n,m)u(m) \qquad \forall n \in T,\quad \forall u \in \mathscr{D}(G).$$

In the above definition, there is no need for $\mathscr{D}(G)$ to have a topology; the convergence of the infinite sum is on the real line.

Of interest is the connection between kernel representation and other properties of map such as linearity, boundedness, and nonanticipation. It is clear that every map that has a kernel representation is weakly linear. However, not all linear systems have a kernel representation. There is an example of a continuous-time linear shift-invariant nonanticipatory system, due to Adam Shefi, in [2, p. 3], that illustrates this point. An example on sequence spaces will be given later. We now examine if boundedness is necessary or sufficient for a linear system to be represented by a kernel. At the level of generality of the above definition for kernel representation, boundedness is not related to kernel representation, because there may not be a topology on $\mathscr{D}(G)$ and $\mathscr{R}(G)$. To examine this relationship we will assume something stronger: we consider systems that are maps from one *normed* space into another. A simple application of the Banach-Steinhaus theorem, e.g. [4], gives the following: Let $G: \mathscr{D}(G) \to l$ be defined by a kernel, with $\mathscr{D}(G)$ a Banach space and $\mathscr{R}(G)$ a normed space. If the family

of projections $\{P_n\}$ is a resolution of the identity on $\mathscr{R}(G)$, then G is bounded. On l_∞, $\{P_n\}$ is not a resolution of identity. If $\mathscr{R}(G) = l_\infty$, using a variant of the Banach-Steinhaus theorem, we can still conclude that G is bounded. While these results are useful, they are not exhaustive because not every kernel-represented map takes a normed space into another. When it does, its domain may not be a Banach space. On the other hand, it is simple to show that if the domain of a bounded linear map has a Schauder basis, then the map has a kernel representation. However, boundedness of the map is an unnecessarily strong requirement: Consider $T : l_\infty(Z_+) \to l_\infty(Z_+)$ defined by $(Tx)(n) = nx(n)$, which clearly unbounded but has a kernel representation.

On the other hand, in functional-analysis literature, bounded linear operators on spaces without a Schauder basis are rarely assumed to be given by a kernel representation. That boundedness is not sufficient for kernel representation is pointed out by an example in [1], with l_∞ as the input and output space. Here is an example that is simpler and sharper but the same in spirit. This example shows that even compactness with discrete spectrum (which is a much stronger condition than boundedness) is not sufficient for kernel representation.

EXAMPLE 2. Consider the space c, the subspace of all converging sequences in $l_\infty(Z_+)$, with l_∞ norm. Fix a nonzero element $y_0 \in c$ such that $\lim_n y_0(n) = 0$. Define $G : c \to c$ by

$$Gx = (\lim x) \cdot y_0.$$

Then G is linear and compact with discrete spectrum $\{0\}$. Its response to an impulse occurring at any time is identically zero. If it is assumed that G has a kernel representation, then the kernel is identically zero, but G is not.

We now consider the relationship between shift invariance, nonanticipation, and kernel representation. The map in the above example is neither shift-invariant nor nonanticipatory. We now construct a map that is linear, shift-invariant, nonanticipatory, and compact, but has no kernel representation.

EXAMPLE 3. Consider the following linear shift-invariant space with l_∞ norm:

$$c_- = \left\{ x \in l_\infty(Z) : \lim_{n \to -\infty} x(n) \text{ exists} \right\}.$$

Let $e \in Z$ be such that $e(n) = 1$ for each $n \in Z$. Define $G : c_- \mapsto c_-$ by

$$Gx = \lim_{n \to -\infty} x(n) \cdot e.$$

Clearly, G is linear, shift-invariant, nonanticipatory, and compact.

A common aspect of both the examples is that the infinite past or the infinite future of the input strongly affects the current output. This motivates the following definition, which will be useful in obtaining necessary and sufficient conditions for kernel representation.

DEFINITION 3.1. A map $G : \mathscr{D}(G) \subseteq l$ is called a *finite-horizon* map if for each $n \in T$ there exist finite integers $l(n), u(n)$ such that $x_l^u = y_l^u$, $x, y \in \mathscr{D}(G)$ implies that $S_n Gx = S_n Gy$.

This means that the current output of a finite-horizon map is completely determined by finite past and finite future of the input. The effect of infinite past and infinite future of the input on the current output is zero. However, the width of the "time window" for the input can depend on time, and need not be uniformly bounded. A (weakly) nonanticipatory map on a unilateral sequence space is an example of a finite-horizon map.

We recall the notion of β-dual of a sequence space. (See, e.g., [3].) Given a sequence space X, its β-dual is given by

$$X^{\beta} := \left\{ y \in l : \left| \sum_{i=t_0}^{\infty} y(i) x(i) \right| < \infty \; \forall x \in X \right\}.$$

It is standard and simple to show that β-dual of l is the space of finitely nonzero sequences. From this follows the next proposition.

PROPOSITION 3.2.

(i) *A map* $G : \mathscr{D}(G) = l \mapsto \mathscr{R}(G) \subseteq l$ *has a kernel representation if and only if it is a weakly linear finite-horizon map.*

(ii) *A map* $G : \mathscr{D}(G) \subseteq l \mapsto \mathscr{R}(G) \subseteq l$ *has a kernel representation if it is a weakly linear finite-horizon map.*

Proof. (i), "only if": Assume that G has a kernel representation. Then the map is weakly linear. Each row of the kernel must belong to the β-dual of l and hence can have only finitely many nonzero entries, implying that the map is of finite horizon.

"If": Fix n. Let $l(n)$ and $u(n)$ be such that $x_l^u = y_l^u$, $x, y \in \mathscr{D}(G)$ implies $S_n Gx = S_n Gy$. Define $J : \mathscr{D}(G) \mapsto \mathbb{R}^{u-l+1}$ by

$$Jx = (x(l), x(l+1), \ldots, x(u-1), x(u)).$$

Let $X := J\mathscr{D}(G)$. For each $x \in X$ let $J^{-1}x$ be the preimage of x under J. Define a map $\tilde{G}_n : X \mapsto \mathbb{R}$ by $\tilde{G}_n x := S_n G J^{-1} x$. Notice that $\tilde{G}_n$ is well defined and is weakly linear. Let $\overline{X} = \operatorname{span} X$. Consider the linear extension $\overline{G}_n : \overline{X} \mapsto \mathbb{R}$ of $\tilde{G}_n$ defined below:

$$\overline{G}_n | X := \tilde{G} \quad \text{and} \quad \overline{G}_n(\alpha_1 x_1 + \cdots + \alpha_k x_k) := \alpha_1 \tilde{G}_n x_1 + \cdots + \alpha_k \tilde{G}_n x_k$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ and for all $x_1, x_2, \ldots, x_k \in X$.

Since $\overline{G}_n$ is a linear functional on a finite-dimensional vector space, it admits the representation

$$\overline{G}_n x = \sum_{i=0}^{u-l} \bar{g}_n(i) x(i) \qquad \forall x \in \overline{X}$$

for some fixed row matrix $\bar{g}_n$. The map G is then given by the kernel defined by

$$g(n,i) := \begin{cases} \bar{g}_n(i-l) & \text{if} \quad l \leqslant i \leqslant u, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) follows from part (i). ∎

It may be noted from the proof that doing analysis locally in time is a key to representation theory. This theme will recur throughout the rest of this section. A weakly linear finite-horizon map has a kernel representation with each row of the kernel having only finitely many nonzero entries. From this proposition and Example 3, it is clear that linearity and shift-invariance of a map on bilateral sequence spaces need not imply that the map is of finite horizon. However, on unilateral spaces with some assumptions on the domains, one may get some useful results. An example is the following proposition. In the proposition, the model for the domain is the space $S^k l(\mathrm{Z}_+)$ for some nonnegative k.

PROPOSITION 3.3. *Let $G : \mathscr{D}(G) \subseteq l(\mathrm{Z}_+) \mapsto l(\mathrm{Z}_+)$ be given. Suppose $\mathscr{D}(G)$ is linear and closed under the family of projections and that there exists an integer $N \in \mathrm{Z}_+$ such that $S^{-1}(I - P_N)\mathscr{D}(G) \subseteq \mathscr{D}(G)$. Under these conditions, if G is linear and weakly shift-invariant, then G is a finite-horizon map.*

REMARK. This proposition tells us that if the domain of a linear shift-invariant map on a unilateral sequence space is sufficiently rich, then the map has a kernel representation. In this proposition, closedness of the domain under the family of projections is part of the hypothesis, unlike in Proposition 2.8. The validity of Proposition 3.3 is the same with or without the canonical embedding. The proof is fairly routine.

From Proposition 3.2 it is clear that the case when $\mathscr{D}(G) \neq l$ is more interesting. In this case, finite-horizon requirement is too strong. Intuitively speaking, the smaller the domain, the easier it should be to obtain a kernel representation. Since finite-horizon maps have a represenation, the next step is to consider maps that are nearly of finite horizon.

DEFINITION 3.4. A map $G: \mathscr{D}(G) \subseteq l \mapsto \mathscr{R}(G) \subseteq l$ is called a *fading-horizon* map if there exists a sequence $G_k : \mathscr{D}(G) \mapsto \mathscr{R}(G_k) \subseteq l$ of finite-horizon maps such that for each $n \in T$

$$S_n Gx = \lim_{k \to \infty} S_n G_k x \qquad \forall x \in \mathscr{D}(G).$$

Again, for this definition it is not necessary for $\mathscr{D}(G)$ to have a topology. Intuitively, the effect of infinite past and infinite future of the input on the current output is vanishingly small for a fading-horizon map. The action of a fading-horizon system at a given time can be approximated by that of a sequence of finite-horizon maps.

In the next definition G is assumed to be linear for simplicity.

DEFINITION 3.5. A linear map $G: \mathscr{D}(G) \subseteq l \mapsto \mathscr{R}(G) \subseteq l$ is called a *strongly fading-horizong* map if $\mathscr{D}(G)$ is a topological space, and if there exists a sequence $G_k : \mathscr{D}(G) \mapsto \mathscr{R}(G_k) \subseteq l$ of finite-horizon maps such that for each $n \in T$, $S_n G_k$ converges to $S_n G$ in the topological dual of $\mathscr{D}(G)$.

Here, the approximation by finite-horizon maps is done locally in time. Since G is linear, for each n, $S_n G, S_n G_k$ are linear functionals, and the convergence is in the space of continuous linear functionals on $\mathscr{D}(G)$. Clearly, a strongly fading-horizon map is a fading-horizon map. The following is easy to prove.

PROPOSITION 3.6.

(*i*) *Let $p \in (1,\infty)$. Then a linear $G: \mathscr{D}(G) \subseteq l_p \mapsto \mathscr{R}(G) \subseteq l$ is a strongly fading-horizon map if and only if it is a fading-horizon map.*

(*ii*) *Let* $p \in [1,\infty)$. *Let* Y *be a normed space. If* $G : \mathscr{D}(G) \subseteq l_p \mapsto \mathscr{R}(G) \subseteq Y$ *is liner and bounded, then* G *is a fading-horizon map.*

Proof. (i): The "only if" direction is obvious.

"If": Fix $p \in (1,\infty)$. Let q be such that $1/p + 1/q = 1$. Then $l_q = l'_p$, the dual of l_p. Fix n. We have $S_nGx = \lim_k S_nG_kx$ for each x. For each k, S_nG_k is in l'_p and S_nG is the weak* limit of S_nG_k. Hence, S_nG is in l'_p (by the Banach-Steinhaus theorem). Therefore, $S_nGx = \Sigma g_n(i)x(i)$ for some g_n in l_q. Since $g_n \in l_q$, it can be approximated in l_q norm by finite-length sequences $\tilde{g}_{n,k}$. For each k, define a finite-horizon map $\tilde{G}_k$ by the kernel $\tilde{g}_k(n,i) := \tilde{g}_{n,k}(i)$. Then $\tilde{G}_k$ is the sequence of finite-horizon maps such that

$$\lim_k \| S_nG - S_n\tilde{G}_k \|_{l_q} = 0.$$

(ii): Fix $p \in [1,\infty)$. Let q be such that $1/p + 1/q = 1$. Fix n. Then, S_nG is a bounded linear functional on l_p and hence is given by the representation $S_nGx = \Sigma g_n(i)x(i)$ for some g_n in l_q. Also, g_n can be approximated by finite-length sequences $g_{n,k}$ such that

$$S_nGx = \lim_k \sum g_{n,k}(i)x(i) \qquad \forall x \in \mathscr{D}(G).$$

For each k, define a finite-horizon map G_k by the kernel $g_k(n,i) := g_{n,k}(i)$. Then G_k is such that $S_nGx = \lim_k S_nG_kx \; \forall x \in \mathscr{D}(G)$. ∎

Clearly, a strongly fading-horizon map (or a finite-horizon map) need not be bounded. The main result of the paper below gives necessary and sufficient conditions for kernel representation on a variety of sequence spaces.

THEOREM 3.7.

(*i*) *Let* $p \in [1,\infty)$. *Let the domain of* $G : \mathscr{D}(G) \subseteq l_p \mapsto \mathscr{R}(G) \subseteq l$ *be a linear space. Then* G *has a kernel representation if and only if* G *is a linear fading-horizon map.*

(*ii*) *Let the domain of* $G : \mathscr{D}(G) \subseteq l_\infty \mapsto \mathscr{R}(G) \subseteq l$ *be a linear space. Then* G *has a kernel representation if and only if* G *is a strongly fading-horizon map.*

Proof. For $p \in [1,\infty)$, let q be such that $1/p + 1/q = 1$.

(i), "if": Fix n. We have $S_nGx = \lim_k S_nG_kx$ for each x. For each k, S_nG_k is in l'_p and S_nG is the weak* limit of S_nG_k. Hence, S_nG is in l'_p (by the Banach-Steinhaus theorem). Therefore, $S_nGx = \Sigma g_n(i)x(i)$ for some g_n in l_q. Set $g(n,i) = g_n(i)$. Then g is a kernel for G.

"Only if": Let the kernel of G be g. Fix n. Then $S_nG = g(n,\cdot)$ is in l_q, since l_q is the β-dual of l_p (e.g., [3]). Hence, $g(n,\cdot)$ can be approximated (in l_q norm if $q \neq \infty$, in weak* topology if $q = \infty$) by finite-length sequences $g_{n,k}$. For each k, define a finite-horizon map G_k by the kernel $g_k(n,i) := g_{n,k}(i)$. Then G_k is the sequence of finite-horizon maps such that $S_nGx = \lim S_nG_kx$ for each x in $\mathscr{D}(G)$.

(ii), "if": Fix n. We have $S_nGx = \lim_k S_nG_kx$ for each x. For each k, S_nG_k is in l_1. Moreover, $\|S_nG - S_nG_k\|_1$ converges as k tends to ∞. Hence S_nG is in l_1.

"Only if": For each n, S_nG is in l_1, since l_1 is the β-dual of l_∞. Therefore S_nG can be approximated in norm by finite-length sequences. ∎

Even when a system has a kernel representation, the representation may not be unique, as shown by the following example.

EXAMPLE 4. Consider $G : l_\infty \to l_\infty$ defined by the following kernel:

$$g = \begin{pmatrix} 1 & 0 & 0 & \cdots & & \\ -\frac{1}{2} & 1 & 0 & 0 & \cdots & \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & \cdots \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

It is easily seen that G is one-to-one. It has a *unique* left-inverse $H : \mathscr{R}(G) \to \mathscr{D}(G)$ such that $HG = I$ on $\mathscr{D}(G)$. It can easily be checked that the following kernel represents the left inverse of G:

$$h_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & & \\ \frac{1}{2} & 1 & 0 & 0 & \cdots & \\ \frac{1}{4} & \frac{1}{2} & 1 & 0 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The following kernel also represents H:

$$h_2 = \begin{pmatrix} 0 & -2 & -4 & -8 & \cdots & \\ 0 & 0 & -2 & -4 & -8 & \cdots \\ 0 & 0 & 0 & -2 & -4 & \cdots \\ 0 & 0 & 0 & 0 & -2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

In fact, each one of the following infinitely many kernels represents H on $\mathscr{R}(G)$:

$$\begin{pmatrix} a_0 & 2(a_0-1) & & 4(a_0-1) & & 8(a_0-1) & \cdots \\ a_1 & 2a_1 & & 2(2a_1-1) & & 4(2a_1-1) & \cdots \\ \vdots & \vdots & \ddots & & \ddots & & \\ a_n & 2a_n & & 2^n a_n & & 2(2^n a_n - 1) & \cdots \\ \vdots & \vdots & & \vdots & & \vdots & \end{pmatrix},$$

where a_i can be freely selected. The kernels h_1, h_2 are special cases of the above form. Clearly, *the* left inverse of G has infinitely many kernel representations.

It is therefore of interest to know when a kernel representation is unique. Uniqueness of kernel representation is related to how rich the domain is. If the domain has enough elements that can distinguish every two infinite matrices in the output, the kernel representation is unique. The following is immediate.

PROPOSITION 3.8. *Suppose $G : \mathscr{D}(G) \subseteq l \mapsto \mathscr{R}(G) \subseteq l$ has a kernel representation. Suppose $\mathscr{D}(G)$ is such that for each $n \in T$, δ_n is in $\mathscr{D}(G)$. Then G has a unique kernel representation.*

If the domain of a map on a unilateral sequence space is not a linear space, then the sufficient condition can be slightly relaxed. It may be noted that the domain of the map in Example 4 violates the sufficient condition in the above proposition. Using Proposition 3.6, Theorem 3.7, and the above proposition, several conclusions can be drawn. Below is an example.

COROLLARY 3.9.

(*i*) *A linear nonanticipatory map $G : \mathscr{D}(G) = l_2(Z_+) \mapsto \mathscr{R}(G) \subseteq l(Z_+)$ has a unique kernel representation.*

(*ii*) *Let $p, r \in [1,\infty)$. A bounded linear map $G : \mathscr{D}(G) = l_p \mapsto \mathscr{R}(G) \subseteq l_r$ has a unique kernel representation.*

4. A SYSTEM AND ITS REPRESENTATION ARE NOT IDENTICAL

When a system can be represented nonuniquely by a kernel, it is of interest to know if properties of the system are reflected in the structure of the kernel. Out of the infinitely many representations for H in the above

example, one is lower-triangular and one is upper-triangular, as shown. Is H, then, nonanticipatory or purely anticipatory? The point is, nonanticipation is a property of a system and is not necessarily a (structural) property of its representation.

We show that H in Example 4 is weakly nonanticipatory [$\mathscr{D}(H) = \mathscr{R}(G)$ is not closed under P_n for any $n > 1$].

CLAIM 1. *H in Example 4 is weakly nonanticipatory.*

Proof. Since H is the inverse of G, we have to show that for all n, $P_nGx_1 = P_nGx_2 \Rightarrow P_nx_1 = P_nx_2$, or equivalently that $P_nx_1 \neq P_nx_2 \Rightarrow P_nGx_1 \neq P_nGx_2$. This follows because for all n, $g(n,n) \neq 0$ (G has direct feedthrough). ∎

We now determine if H is shift-invariant. Observe that the domain of H in Example 4 is shift-invariant. That H is shift-invariant follows from Proposition 2.7. However, H has some kernel representations which have Toeplitz structure (constant along the diagonals), and some which do not. Shift invariance is clearly a property of a system that may or may not be reflected in the structure of its representation.

We now point out that boundedness of a map may not be reflected in the structure of its representation.

EXAMPLE 3. Consider $G : l_\infty(Z_+) \to l_\infty(Z_+)$ defined by the following kernel:

$$g = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & \cdots & & \\ 1 & -\frac{1}{2} & 0 & 0 & \cdots & \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

It is simple to show that the left inverse of G exists and is bounded, since for each $x \in l_\infty$ we have $\|Gx\|_\infty \geqslant \frac{1}{2}\|x\|_\infty$. One kernel representation of G^{-1} is

$$\begin{pmatrix} -2 & 0 & 0 & \cdots & & \\ -4 & -2 & 0 & 0 & \cdots & \\ -8 & -4 & -2 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

However, that G^{-1} is bounded is not apparent from the above representation (or from a minimal state-space representation of G^{-1}: $A = 2$, $B = -1$, $C = 4$, $D = -2$).

Naturally, from an infinite-matrix kernel representation alone, it is not clear what the domain of the underlying system is. An obvious linear extension to an l_p space may not carry over properties of the system such as shift invariance and nonanticipation.

From the examples, we conclude that properties such as shift invariance, nonanticipation, and boundedness are properties of a system and are not necessarily structural properties of a representation of the system (unless the representation is unique). That is, a system is a logically distinct object from its representation. It is the behavior (graph) of the system that needs to be examined for properties of interest, and not the structure of a representation of the system. The behavior of a system is more fundamental than a representation of the system [6].

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Performance Robustness of Discrete-time Systems with Structured Uncertainty *

Mustafa Khammash
Dept. of Elect. and Comp. Eng.
Iowa State University
Ames, Iowa

J.B. Pearson
Dept. of Elec. and Comp. Eng.
Rice University
P.O.Box 1892
Houston, TX 77251-1892

Abstract

Given an interconnection of a nominal discrete-time plant and a stabilizing controller together with structured, norm bounded, nonlinear/time-varying perturbations, necessary and sufficient conditions for robust stability and performance of the system are provided. This is done by first showing that performance robustness is equivalent to stability robustness in the sense that both problems can be dealt with in the framework of a general stability robustness problem. The resulting stability robustness problem is next shown to be equivalent to a simple algebraic one, the solution of which provides the desired necessary and sufficient conditions for performance/stability robustness. These conditions provide an effective tool for robustness analysis and can be applied to a large class of problems. In particular, it is shown that some known results can be obtained immediately as special cases of these conditions.

1 Introduction

For systems with bounded energy signals, the $\mathcal{H}^\infty$ norm is the most suitable norm to use. When dealing with robust performance in the context of linear feedback systems with $\mathcal{H}^\infty$ norm performance objectives, the paper by Doyle [3] introduces a nonconservative measure of performance for linear feedback systems in the presence of structured model uncertainties. This approach is based on a matrix function called the Structured Singular Value, where stability and performance robustness are dealt with in the same framework. The class of perturbations treated are linear time-invariant norm bounded perturbations.

When the system at hand does not involve bounded energy signals but rather bounded magnitude signals as is the case when bounded persistent disturbances are present, the more suitable norm is the $\mathcal{A}$ norm or ℓ^1 norm. In [4]/[5] Dahleh and Pearson provided a complete solution to the problem of minimizing the $\mathcal{A}$ norm of a linear time-invariant continuous/discrete-time system through the choice of a stabilizing controller. The optimal controllers obtained in the discrete time case are more useful than those in the continuous time case since they are easier to implement physically.

In this paper, we present a solution to the robustness problem in the ℓ^1 setting. The class of perturbations considered consists of norm bounded perturbations allowed to be time-varying or nonlinear. We provide necessary and sufficient conditions for stability robustness for structured perturbations where any number of perturbations can enter between any two points in the system. In addition, we allow performance objectives to be considered and provide necessary and sufficient conditions for these objectives to be achieved in a robust manner subject to robust stability. This is done by showing that the stability and performance robustness problem is equivalent to a simple algebraic problem which can be easily solved to give the desired nonconservative conditions for stability and performance robustness. We show how the results in [6] and in [7] can be obtained as special cases of this theory.

2 Notation

$\mathbb{R}^+$ Nonnegative real numbers.

ℓ^∞ Space of all bounded sequences of real

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numbers, i.e. $x = \{x(k)\}_{k=0}^{\infty} \in \ell^{\infty}$ if and only if $\sup_k |x(k)| < \infty$. If $x \in \ell^{\infty}$ then $\|x\|_{\infty} = \sup_k |x(k)|$.

ℓ^{∞}_q Space of q-tuples of elements of ℓ^{∞}. If $x = (x_1, \ldots, x_q) \in \ell^{\infty}_q$ then $\|x\|_{\infty} = \max_i \|x_i\|_{\infty}$.

ℓ^1 Space of absolutely summable sequences. If $x \in \ell^1$ then $\|x\|_1 = \sum_{k=0}^{\infty} |x(k)| < \infty$.

$\ell^1_{p\times q}$ Space of $p \times q$ matrices with entries in ℓ^1. If $x = (x_{ij}) \in \ell^1_{p\times q}$, then $\|x\|_1 := \max_{1\leq i\leq p} \sum_{j=1}^{q} \|x_{ij}\|_1$.

$\mathcal{L}^{p\times q}_{TV}$ The space of all bounded linear causal operators mapping ℓ^{∞}_q to ℓ^{∞}_p. If $R \in \mathcal{L}^{p\times q}_{TV}$ then $\|R\| := \sup_{x\neq 0} \frac{\|Rx\|_{\infty}}{\|x\|_{\infty}}$ which is the induced operator norm. Each R in $\mathcal{L}^{p\times q}_{TV}$ can be completely characterized by its block lower-triangular pulse response matrix.

$\mathcal{L}^{p\times q}_{TI}$ Subspace of $\mathcal{L}^{p\times q}_{TV}$ consisting of time-invariant operators. For each $R \in \mathcal{L}^{p\times q}_{TI}$ corresponds a unique r in $\ell^1_{p\times q}$ where r_{ij} is the impulse response of R_{ij}, the component of R mapping the the jth input to the ith output. The induced operator norm of R as a map from ℓ^{∞}_q to ℓ^{∞}_p is equal to the norm of r in $\ell^1_{p\times q}$, which we shall also refer to as the $\mathcal{A}$ norm.

Problem Setup

We are mainly interested in ℓ^{∞} signals and discrete-time systems. Aside from that, the only conditions imposed will be those needed to guarantee the well-posedness of the problem. Common to all the problems in which stability and performance of a certain system are to be studied under the effect of perturbations are a nominal plant and a controller stabilizing it. In our case, both of these are assumed to be linear time-invariant discrete-time systems. There is no reason why only one nominal plant or controller can be considered, and so as many as desired can be incorporated as long as the resulting nominal system is stable. As for the perturbations, they are first modeled as strictly causal linear maps

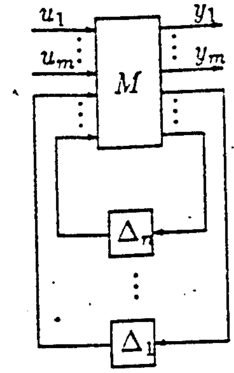

Figure 1: Stability and Performance Robustness Problem

taking ℓ^{∞} signals to ℓ^{∞} signals with bounded induced norms. Hence the perturbations are allowed to be time-varying. Nonlinear perturbations are treated in section 6. There can be as many perturbations as desired and they can enter anywhere in the system. So for a specific set of bounds on the norms of the perturbations, we have a family of systems each of which is composed of the nominal part and a set of fixed perturbations with norms less than the corresponding given bounds. The first objective is to determine when every member of that class of systems is stable, i.e. when our system is robustly stable. In many cases, stability is not all that is required from a system, and certain performance objectives are to be met. A useful and popular objective is keeping small the norm of the function mapping an external input, say u, to a certain signal in the loop, call it y. Since there could be more than one such objective, let us denote the resulting functions by $T_{y_iu_i}$ for $i = 1, \ldots, m$, where $T_{y_iu_i}$ is the function mapping signals at point u_i to signals at point y_i. Because we are mainly concerned with ℓ^{∞} signals, the norm we want to be small would be in our case the induced ℓ^{∞} norm. Now our objective is to determine, given a set of m positive real numbers $\gamma_1, \ldots, \gamma_m$, conditions under which our system is stable and satisfies $\|T_{y_iu_i}\| < \gamma_i$ for all allowable perturbations. In other words, when does our system achieve robust performance?

We now formally set up the stability and performance robustness problem mentioned above. Figure 1 represents a quite general configuration appropriate for describing problems with uncertainty. In the figure, M represents the interconnection of the nominal plant and the stabilizing controller, and is therefore linear, time-invariant, and stable. Each Δ_i represents the perturbations between two points in the system, and has norm less than or equal to one. Of course there is no loss of generality in assuming that the chosen bound on the norms of each

of the Δ_i's is one, since any other set of numbers could be absorbed in M. We will restrict the Δ_i's to be *strictly* causal in order to guarantee the well posedness of the system. This is not a serious restriction and can be removed if it is known that the perturbation/nominal system connection is well-posed. Accordingly we can define the classes of perturbations to which the $\Delta_i's$ belong. Assuming the perturbations enter at n places, and that each has p_i inputs and q_i outputs we have $\Delta_i \in \Delta(p_i, q_i)$ where $\Delta(p_i, q_i) := \left\{\Delta \in \mathcal{L}_{TV}^{p_i \times q_i} : \Delta \text{ is strictly causal and } \|\Delta\| \leq 1\right\} \quad i = 1, \ldots, n.$

Note that Δ_i is not dependent in any way on Δ_j when $j \neq i$. The only restriction is that Δ_i belongs to $\Delta(p_i, q_i)$ for each i. Next let $p = \sum_i p_i$, and $q = \sum_i q_i$. By $\mathcal{D}[(p_1, q_1); \cdots; (p_n, q_n)]$ we mean the set of all operators mapping ℓ_q^∞ to ℓ_p^∞ of the form:

$$D = diag(\Delta_1, \ldots, \Delta_n),$$

where Δ_i belongs to $\Delta(p_i, q_i)$. When the pairs (p_i, q_i) are known, they will be dropped from the notation and $\mathcal{D}$ will be understood to mean the above set. We will say the system in fig. 1 achieves robust stability if the system is stable for all $D \in \mathcal{D}[(p_1, q_1); \ldots; (p_n, q_n)]$. We will say it achieves robust performance if it achieves robust stability and $\|T_{y_i u_i}\| < 1$ for all i and for all D in $\mathcal{D}[(p_1, q_1); \ldots; (p_n, q_n)]$.

In the context of this setup, our problem can be stated as follows:

Problem Statement. *Find necessary and sufficient conditions for the system in fig. 1 to achieve robust performance.*

4 Performance Robustness vs. Stability Robustness

In this section, we provide a theorem establishing a relation between stability robustness and performance robustness. It states that performance robustness in one system is equivalent to stability robustness in another one formed by adding a fictitious perturbation. A similar result has been shown to hold in [9] when the perturbations are linear time-invariant and when the 2-norm is used to characterize the perturbation class. The usefulness of this theorem stems from the fact that we can now concentrate on finding conditions for achieving stability robustness alone. Once we do, performance robustness comes for free.

Consider the two systems shown in fig. 2, where $M \in \mathcal{L}_{TI}^{q \times p}$ and $\Delta_i \in \Delta(p_i, q_i)$. In system

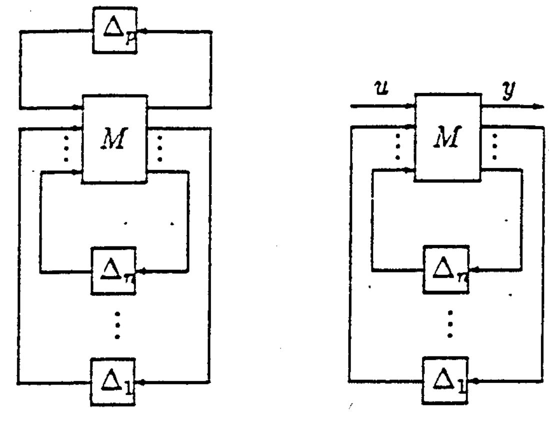

Figure 2: Equivalence of Stability and Performance Robustness

II, u is a vector input of size $\bar{p}$ and y is an output vector of size $\bar{q}$. In system I, $\Delta_p \in \Delta(\bar{p}, \bar{q})$. It follows that $p = \bar{p} + \sum_i p_i$ and $q = \bar{q} + \sum_i q_i$. Subdivide M in the following manner:

$$M = \begin{pmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{pmatrix}$$

where $\tilde{M}_{11} \in \mathcal{L}_{TI}^{\bar{q} \times \bar{p}}$.

We now state the following theorem establishing the relation between System I and System II.

Theorem 1. *The following four statements are equivalent:*

i) System I achieves robust stability.

ii) $(I - M\tilde{D})^{-1}$ *is* ℓ^∞*-stable for all* $\tilde{D} \in \mathcal{D}[(\bar{p}, \bar{q}); (p_1, q_1); \cdots; (p_n, q_n)]$.

iii) $(I - \tilde{M}_{22}D)^{-1}$ *is* ℓ^∞*-stable and* $\|\tilde{M}_{11} + \tilde{M}_{12}D(I - \tilde{M}_{22}D)^{-1}\tilde{M}_{21}\| < 1$, *for all* D *belonging to* $\mathcal{D}[(p_1, q_1); \ldots; (p_n, q_n)]$.

iv) System II achieves robust performance.

5 Conditions for Stability Robustness

It has been shown in the previous section that we can convert a performance robustness problem into one which involves stability robustness alone. We can therefore concentrate only on stability robustness. We seek nonconservative conditions for achieving stability robustness which are easy to verify. Before we begin, we establish some notational conventions. Throughout this section, the perturbation set will be $\mathcal{D}[(p_1, q_1); \ldots; (p_n, q_n)]$ for some positive integers

$p_1, \ldots, p_n$ and $q_1, \ldots, q_n$. M belongs to $\mathcal{L}_{TI}^{q\times p}$ where $p := \sum_i p_i$ and $q := \sum_i q_i$. Hence M can be partitioned as follows:

$$M = \begin{pmatrix} M_{11} & \ldots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \ldots & M_{nn} \end{pmatrix}$$

where M_{ij} has size $q_i \times p_i$. Next, we will state our main result establishing the equivalence of the stability robustness problem to a simple algebraic one. Depending on the region in which this algebraic problem has its solutions, we can conclude whether or not our system achieves robust stability, and by the results of the previous section, robust performance. In order not to clutter the exposition, we first state and prove this theorem in the scalar case. Hence $p_1 = \cdots = p_n = q_1 = \cdots = q_n = 1$.

Theorem 2. *$(I - MD)^{-1}$ is not ℓ^∞- stable for some $D \in \mathcal{D}[(1,1);\cdots;(1,1)]$ if and only if the system*

$$x_i \leq \sum_{j=1}^{n} \|M_{ij}\|_{\mathcal{A}} x_j \qquad i = 1, \ldots, n$$

has a solution $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ in $(\mathbb{R}^+)^n \backslash \{0\}$.

With this theorem, our problem stated in section 3 is essentially solved. Applying this theorem to the performance and stability robustness problem stated earlier, reduces it to a simple algebraic one in which the object is to determine whether a certain system of inequalities has a solution in a particular region in $\mathbb{R}^n$. What makes this algebraic problem particularly attractive is that the set of inequalities that arises relates in a simple and direct manner to the original problem. Only norms of the subentries of the M matrix arise, and they do so in the same general order that they do in M. The question that arises naturally at this point is how can one determine whether the system of inequalities at hand has a solution in the related region of $\mathbb{R}^n$? It turns out, that no search techniques are needed to accomplish this task and the answer to this question can be determined by evaluating certain expressions directly. These expressions also involve norms of the subentries of M and thus are easy to compute. The derivation of these alternate conditions for stability and performance robustness is the next topic of discussion.

The first step in restating the conditions involving the set of inequalities is to make the following observation:

Observation. *The system of inequalities:*

$$x_i \leq \sum_{j=1}^{n} \|M_{ij}\|_{\mathcal{A}} x_j \qquad i = 1, \ldots, n$$

has a solution in $(\mathbb{R}^+)^n \backslash \{0\}$ if and only if either $\|M_{nn}\|_{\mathcal{A}} \geq 1$ or $\|M_{nn}\|_{\mathcal{A}} < 1$ and the system of inequalities:

$$x_i \leq \sum_{j=1}^{n-1} \left(\|M_{ij}\|_{\mathcal{A}} + \frac{\|M_{in}\|_{\mathcal{A}} \|M_{nj}\|_{\mathcal{A}}}{1 - \|M_{nn}\|_{\mathcal{A}}} \right) x_j \qquad i = 1, \ldots, n$$

has a solution in $(\mathbb{R}^+)^{n-1} \backslash \{0\}$.

Notice that this observation allows us to replace the task of determining whether any solutions to a set of n inequalities lie in a certain region by the simpler one of determining whether the solutions to a set of $n-1$ inequalities lie in a smaller region together with a simple test on the norm of M_{nn}. It is easily seen how this can be repeated until we completely replace all such conditions by tests on expressions involving norms of the M_{ij}'s, a much simpler task. Table 1 lists some of these for a few values of n.

In order to discuss the multivariable case we will need to make reference to the rows of M_{ij} which are themselves stable rational functions. Let us denote the mth row of M_{ij} by $(M_{ij})_m$. Since we will no longer restrict the p_i's and q_i's to be equal to one, the following set is not necessarily a singleton:

$$\mathcal{K} := \{(k_1, \ldots, k_n) \in Z^n : 1 \leq k_i \leq q_i\}.$$

From this definition it is clear that the set $\mathcal{K}$ has exactly $\prod_{i=1}^{n} q_i$ elements. To each $k \in \mathcal{K}$ corresponds the system of inequalities: $x_i \leq \sum_{j=1}^{n} \|(M_{ij})_{k_i}\|_{\mathcal{A}} x_j$ where $k = (k_1, \ldots, k_n)$. As the next theorem shows, it is the solutions of these inequalities that are of concern when seeking necessary and sufficient conditions for stability and performance robustness in the MIMO case.

Theorem 3. *$(I - MD)^{-1}$ is not ℓ^∞- stable for some $D \in \mathcal{D}[(p_1, q_1);\ldots;(p_n, q_n)]$ if and only if for some $k = (k_1, \ldots, k_n) \in \mathcal{K}$, the system*

$$x_i \leq \sum_{j=1}^{n} \|(M_{ij})_{k_i}\|_{\mathcal{A}} x_j \qquad i = 1, \ldots, n$$

has a solution $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ in $(\mathbb{R}^+)^n \backslash \{0\}$.

n	Necessary and Sufficient Conditions for Stability Robustness
1	$\lVert M\rVert_{\mathcal{A}} < 1$
2	$\lVert M_{22}\rVert_{\mathcal{A}} < 1$ $\lVert M_{11}\rVert_{\mathcal{A}} + \dfrac{\lVert M_{12}\rVert_{\mathcal{A}}\lVert M_{21}\rVert_{\mathcal{A}}}{1-\lVert M_{22}\rVert_{\mathcal{A}}} < 1$
3	$\lVert M_{33}\rVert_{\mathcal{A}} < 1$ $\lVert M_{22}\rVert_{\mathcal{A}} + \dfrac{\lVert M_{23}\rVert_{\mathcal{A}}\lVert M_{32}\rVert_{\mathcal{A}}}{1-\lVert M_{33}\rVert_{\mathcal{A}}} < 1$ $\lVert M_{11}\rVert_{\mathcal{A}} + \dfrac{\lVert M_{13}\rVert_{\mathcal{A}}\lVert M_{31}\rVert_{\mathcal{A}}}{1-\lVert M_{33}\rVert_{\mathcal{A}}} + \dfrac{\left(\lVert M_{12}\rVert_{\mathcal{A}} + \dfrac{\lVert M_{13}\rVert_{\mathcal{A}}\lVert M_{32}\rVert_{\mathcal{A}}}{1-\lVert M_{33}\rVert_{\mathcal{A}}}\right)\left(\lVert M_{21}\rVert_{\mathcal{A}} + \dfrac{\lVert M_{23}\rVert_{\mathcal{A}}\lVert M_{31}\rVert_{\mathcal{A}}}{1-\lVert M_{33}\rVert_{\mathcal{A}}}\right)}{1-\left(\lVert M_{22}\rVert_{\mathcal{A}} + \dfrac{\lVert M_{23}\rVert_{\mathcal{A}}\lVert M_{32}\rVert_{\mathcal{A}}}{1-\lVert M_{33}\rVert_{\mathcal{A}}}\right)} < 1$

Table 1: Conditions for Stability/Performance Robustness for $n = 1, 2,$ and 3

6 Nonlinear Perturbations

In this section, it will be shown that if the class of perturbations is enlarged to include norm-bounded nonlinear perturbations, then the conditions for robust stability remain the same. This means that robustness to linear time-varying perturbations will automatically guarantee robustness to nonlinear perturbations as well. Furthermore, it is shown that when enlarging the perturbation class to include nonlinear perturbations, stability robustness remains equivalent to performance robustness, and so the conditions for stability and performance robustness for time-varying perturbations are the same as those for nonlinear perturbations. For simplicity we shall consider the scalar case here. We start by extending our definition for the perturbation class to include nonlinear perturbation. So define

$$\mathcal{D}_{NL}[(p_1,q_1);\ldots;(p_n,q_n)] := \Big\{\mathrm{diag}(\Delta_1,\ldots,\Delta_n) : \Delta_i \text{ is strictly causal and} \sup_{x\neq 0}\frac{\lVert\Delta_i x\rVert_\infty}{\lVert x\rVert_\infty} \le 1\Big\}.$$

For simplicity we adopt the following notation:

$$\mathcal{D}(n) := \mathcal{D}[\overbrace{(1,1);\ldots;(1,1)}^{n}]$$

$$\mathcal{D}_{NL}(n) := \mathcal{D}_{NL}[\overbrace{(1,1);\ldots;(1,1)}^{n}]$$

Theorem 4. *$(I - MD)^{-1}$ is ℓ^∞- stable for all $D \in \mathcal{D}(n)$ if and only if it is ℓ^∞- stable for all $D \in \mathcal{D}_{NL}(n)$.*

We have shown that stability robustness is equivalent to performance robustness when the class of perturbations is $\mathcal{D}(n)$. It does not immediately follow that this should be true if the perturbation class were $\mathcal{D}_{NL}(n)$. Next we show that indeed stability robustness is equivalent to performance robustness even when enlarging the perturbation class to include nonlinear perturbations.

We will assume the class of perturbations is $\mathcal{D}_{NL}(n)$ and that we have one performance objective consisting of keeping the norm of the function mapping the external input u to the output y less than one. (Figure 2, SYSTEM II).

Theorem 5. *$(I - \tilde{M}_{22}D)^{-1}$ is ℓ^∞- stable and $\lVert\tilde{M}_{11} + \tilde{M}_{12}D(I-\tilde{M}_{22}D)^{-1}\tilde{M}_{21}\rVert < 1$ for all $D \in \mathcal{D}_{NL}(n)$ if and only if $(I - M\tilde{D})^{-1}$ is ℓ^∞- stable for all $\tilde{D} \in \mathcal{D}_{NL}(n+1)$.*

7 Some Applications

7.1 Stability Robustness (Unstructured Pert.)

This is the simplest case. The perturbations take the form of one Δ having q inputs and p outputs. The question then is when is $(I - M\Delta)^{-1}$ stable

for all Δ in $\Delta(p,q)$? Equivalently, when is the interconnection of $M \in \mathcal{L}_{TI}^{q\times p}$ and Δ stable for all Δ in $\Delta(p,q)$? From Theorem 3, a necessary and sufficient condition for robust stability is that none of the q inequalities:

$$x \leq \|(M)_i\|_{\mathcal{A}} \cdot x \qquad i = 1,\ldots,q$$

has a solution in $(0,\infty)$. Clearly, a necessary and sufficient condition for that to happen is that $\|(M)_i\|_{\mathcal{A}} < 1$ for all i, or equivalently $\|M\|_{\mathcal{A}} < 1$. This is exactly the problem solved by Dahleh and Ohta in [6].

7.2 Input Sensitivity in the Presence of Multiplicative Input Perturbations

Let P_o be a given nominal linear shift-invariant discrete-time plant with q inputs and p outputs. Consider the following family of plants formed by adding weighted multiplicative perturbations to this nominal plant:

$$\Pi := \{P : P = P_o(I + W_1\Delta), \Delta \in \Delta(q,q)\}$$

where $W_1 \in \mathcal{L}_{TI}^{q\times q}$. Let $S(P_o)$ be defined as follows:

$$S(P_o) := \{C : C \text{ is linear causal shift-invariant controller stabilizing } P_o\}$$

For a fixed $C \in S(P_o)$ and $\gamma > 0$ we will now obtain necessary and sufficient conditions for C to stabilize every $P \in \Pi$, and at the same time satisfy $\|(I+CP)^{-1}W_2\| < \gamma$ for all P in Π. Hence the performance objective in this case is keeping small the norm of the weighted input sensitivity function $(I+CP)^{-1}W_2$ despite the presence of the multiplicative perturbations.

This problem can be set up in the framework discussed in the previous sections where a fictitious perturbation replaces the performance objective, thus transforming this stability and performance robustness problem into a stability robustness problem alone. This alternate problem has $\mathcal{D}[(q,q),(q,q)]$ as the class of perturbations, and an M matrix of the following form:

$$M = \begin{pmatrix} \frac{1}{\gamma}(I+CP_o)^{-1}W_2 & CP_o(I+CP_o)^{-1}W_1 \\ \frac{1}{\gamma}(I+CP_o)^{-1}W_2 & CP_o(I+CP_o)^{-1}W_1 \end{pmatrix}$$

From table 1 and Theorem 3, necessary and sufficient conditions for robust stability for this problem, and hence for robust performance for the original one are:

- $\|(T_o)_i\|_{\mathcal{A}} < 1 \qquad i = 1,\ldots,q.$
- $\|\frac{1}{\gamma}(S_o)_i\|_{\mathcal{A}} + \dfrac{\|(T_o)_i\|_{\mathcal{A}}\|\frac{1}{\gamma}(S_o)_j\|_{\mathcal{A}}}{1-\|(T_o)_j\|_{\mathcal{A}}} < 1$
  $i,j = 1,\ldots,q.$

where $S_o := (I+CP_o)^{-1}W_2$, and $T_o := CP_o(I+CP)^{-1}{}_oW_1$. Equivalently, these conditions can be written as:

- $\|T_o\|_{\mathcal{A}} < 1.$
- $\max_{1\leq i\leq q} \dfrac{\|(S_o)_i\|_{\mathcal{A}}}{1-\|(T_o)_i\|_{\mathcal{A}}} < \gamma.$

If we define $\Psi := \{C \in S(P_o) : C \text{ stabilizes all } P \in \Pi\}$, then it follows from our stability robustness conditions for one Δ that $C \in \Psi$ if and only if $C \in S(P_o)$ and $\|T_o\|_{\mathcal{A}} < 1$. Hence we have shown through the two conditions obtained above that for any $C \in \Psi$

$$\sup_{P\in\Pi} \|(I+CP)^{-1}W_2\| = \max_i \frac{\|(S_o)_i\|_{\mathcal{A}}}{1-\|(T_o)_i\|_{\mathcal{A}}}.$$

This is exactly the result obtained by the authors in [7] using a different approach. In fact, it is not difficult to show [7] that for any $\gamma > 0$

$$C \in \Psi \text{ and } \sup_{P\in\Pi}\|(I+CP)^{-1}W_2\| < \gamma \qquad \text{iff}$$
$$C \in S(P_o) \text{ and } \|(S_o \quad \gamma T_o)\|_{\mathcal{A}} < \gamma.$$

Since it is known [4,10,11] how to solve problems like

$$\min_{C\in S(P_o)} \|(S_o \quad \gamma T_o)\|_{\mathcal{A}}$$

it is clear how an iterative scheme can be devised whereby the value of γ can be increased or decreased according to the outcome of the optimization problem stated above, until γ is as close as desired to γ_{opt}, where

$$\gamma_{opt} := \inf_{C\in\Psi}\sup_{P\in\Pi}\|(I+CP)^{-1}W_2\|.$$

Since at each iteration step a controller that achieves the minimum can be computed, we can find a controller that achieves arbitrarily closely γ_{opt}.

7.3 Output Sensitivity in the Presence of Output Multiplicative Perturbations

For this case let

$$\Pi := \{P : P = (I + \Delta W_1)P_o, \Delta \in \Delta(q,q)\}$$

where P_o and W_1 are as before. Suppose we are now interested in the norm of the output sensitivity function as a performance measure. For $C \in S(P_o)$, the M matrix now has the form

$$M = \begin{pmatrix} \frac{1}{\gamma}W_2(I+P_oC)^{-1} & \frac{1}{\gamma}W_2(I+P_oC)^{-1} \\ W_1P_oC(I+P_oC)^{-1} & W_1P_oC(I+P_oC)^{-1} \end{pmatrix}$$

Hence, from table 1 necessary and sufficient conditions for robust stability and performance are now:

- $\|(T_o)_i\|_{\mathcal{A}} < 1 \qquad i = 1, \ldots, q.$
- $\|\frac{1}{\gamma}(S_o)_i\|_{\mathcal{A}} + \dfrac{\|(T_o)_j\|_{\mathcal{A}}\|\frac{1}{\gamma}(S_o)_i\|_{\mathcal{A}}}{1-\|(T_o)_j\|_{\mathcal{A}}} < 1$
  $i, j = 1, \ldots, q.$

where $T_o := W_1P_oC(I+P_oC)^{-1}$ and $S_o := W_2(I+P_oC)^{-1}$. Equivalently, these conditions can be written as follows:

- $\|T_o\|_{\mathcal{A}} < 1.$
- $\dfrac{\|S_o\|_{\mathcal{A}}}{1-\|T_o\|_{\mathcal{A}}} < \gamma.$

With Ψ defined as before, it follows that foror any $C \in \Psi$,

$$\sup_{P\in\Pi} \|W_2(I+PC)^{-1}\| = \frac{\|S_o\|_{\mathcal{A}}}{1-\|T_o\|_{\mathcal{A}}}.$$

Even though these conditions are different from those obtained in the input sensitivity case, for a scalar plant they are actually the same.

8 Conclusion

We have provided in the previous sections necessary and sufficient conditions for achieving sta bility and performance robustness. These conditions can be applied to a large class of problems in which multiple perturbations can enter in various configurations. The conditions involve no more that computing the $\mathcal{A}$ norm of certain transfer functions, a task which can be done to any degree of accuracy with relative ease. Consequently, these conditions provide a particularly attractive method for the analysis of stability and performance robustness. We have also shown that in some important cases obtaining a controller with optimal robustness properties can be done through a simple iterative scheme. Synthesis of controllers in the more general case, is an interesting problem which is currently under research.

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Robustness Synthesis for Discrete-time Systems with Structured Uncertainty *

Mustafa Khammash
Electrical Engineering & Computer Engineering Dept.
Iowa State University
Ames, Iowa 50011

J.B. Pearson
Department of Electrical & Computer Engineering
Rice University
Houston, TX 77251–1892

Abstract

Necessary and sufficient conditions for stability and performance robustness of discrete-time systems are provided in terms of the spectral radius of a certain nonnegative matrix. The conditions are easily computable and provide a simple method to do synthesis of robust controllers via an iteration scheme which utilizes the properties of the spectral radius.

1 Introduction

In [1,2,3], necessary and sufficient conditions were derived for stability robustness when structured ℓ^∞ norm-bounded perturbations were assumed. These conditions were given in terms of the region in which a system of inequalities has its solution. The system of inequalities is completely determined by the interconnection of the nominal system at hand and stabilizing controller. Even though conditions for stability robustness are important in their own right, they also give conditions for performance robustness. This has been demonstrated in [1,2] where it was shown that a performance robustness problem can be converted to a stability robustness problem by adding a fictitious perturbation block to represent the performance. The conditions for stability robustness which result are exactly those for performance robustness for the original problem.

In this paper we establish a connection between the conditions for stability robustness and the spectral radius of a certain nonnegative matrix. Use of the spectral radius conditions allows us not only to obtain numerically efficient ways for determining when a certain system achieves robust stability and performance, but it also provides us with the means to design controllers which provide suboptimal robustness properties.

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2 Notation

$\mathbb{R}^+$ Nonnegative real numbers.

ℓ^∞ Space of all bounded sequences of real numbers, i.e. $x = \{x(k)\}_{k=0}^{\infty} \in \ell^\infty$ if and only if $\sup_k |x(k)| < \infty$. If $x \in \ell^\infty$ then $\|x\|_\infty = \sup_k |x(k)|$.

ℓ^1 Space of absolutely summable sequences. If $x \in \ell^1$ then $\|x\|_1 = \sum_{k=0}^{\infty} |x(k)| < \infty$.

$\|\cdot\|_{\mathcal{A}}$ The $\mathcal{A}$ norm of a z-transform of an ℓ^1 sequence, is the ℓ^1 norm of that sequence. So for an LTI system, this will be the ℓ^1 norm of the pulse response of that system. This is a measure of the maximum amplitude gain of the system. For a system matrix, the $\mathcal{A}$-norm is the maximum row sum of individual SISO entry norms.

Δ The set of all operators mapping ℓ^∞ to itself, with induced ℓ^∞ norm less than or equal to one. Hence, $\Delta := \left\{\Delta : \sup_{x\neq 0} \frac{\|\Delta x\|_\infty}{\|x\|_\infty} \leq 1\right\}$.

$\mathcal{D}(n)$ The set of all diagonal operators of the form $D = diag(\Delta_1, \ldots, \Delta_n)$ where $\Delta_i \in \Delta$.

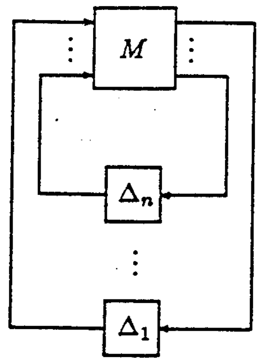

Figure 1: Stability Robustness Problem

3 Setup

We start by setting up the stability robustness problem. Given is an interconnection of linear time-invariant plant/plants and linear time-invariant controller together with n perturbation blocks, say, $\Delta_1, \ldots, \Delta_n$. These blocks represent the system uncertainty which is assumed to take place in n different locations in the interconnection. Each perturbation block, Δ_i belongs to the class Δ and is therefore norm bounded. The Δ_i's are independent of each other reflecting the situation when the uncertainty has different sources. Next, let M denote that part of the interconnection which includes the nominal plant and stabilizing controller. M will have n inputs and n outputs corresponding to the interconnection with the pertrubation blocks.

Whereas M is given and fixed (at least in the analysis problem where a controller is given), each perturbation block, Δ_i is allowed to vary over the set Δ. The combined effect of all perturbation blocks can be equivalently captured by one peturbation block , D, which has a diagonal structure. D now belongs to the class $\mathcal{D}(n)$. With this setup in mind, the system is said to achieve robust stability if it is ℓ^∞-stable for all $D \in \mathcal{D}(n)$. The next section is concerned with various necessary and sufficient conditions for the system in fig. 1 to achieve robust stability. Some of these conditions will prove useful in the synthesis of controllers with suboptimal robustness properties.

4 Main Results

In this section, we state without proof our main theorem establishing the necessary and sufficient conditions for robust stability of the system in fig. 1 in terms of the spectral radius of a certain matrix as well as other conditions. We start by defining $|M|$. Since M is linear time-invariant and stable with n inputs and outputs M_{ij}, the map taking the jth input to the ith output has a pulse response which belongs to the space ℓ^1. The ℓ^1 norm, or the $\mathcal{A}$ norm of M_{ij} can be computed arbitrarily accurately. We define $|M|$ to be the following matrix of norms

$$|M| := \begin{pmatrix} \|M_{11}\|_{\mathcal{A}} & \cdots & \|M_{1n}\|_{\mathcal{A}} \\ \vdots & & \vdots \\ \|M_{n1}\|_{\mathcal{A}} & \cdots & \|M_{nn}\|_{\mathcal{A}} \end{pmatrix}.$$

Defining $\mathcal{R}$ to be the set of all $n \times n$ real diagonal matrices with positive entries on the diagonal, we can state the following theorem:

Theorem 1 *The following are all equivalent:*

1. *The system in fig. 1 achieves robust stability.*
2. *The system of inequalities:*

$$x_i \leq \sum_{j=1}^{n} \|M_{ij}\|_{\mathcal{A}} x_j \qquad i = 1, \ldots, n$$

has no solutions in $(\mathbb{R}^+)^n \backslash \{0\}$.

3. $\rho(|M|) < 1$, *where* $\rho(|M|)$ *denotes the spectral radius of* $|M|$.
4. $\inf_{R \in \mathcal{R}} \|R^{-1} M R\|_{\mathcal{A}} < 1$.

That *1* and *2* are equivalent has been shown in [1,2]. The important equivalence for the purposes of this paper is that of *1* and *4* since this allows us to do controller synthesis as will be discussed next.

Since M forms the interconnection of the nominal linear time invariant system and linear time-invariant controller it can be put in the following form:

$$M = T_1 - T_2 Q T_3$$

where T_1, T_2, and T_3 are stable and depend only on the nominal plant. Q, is a free parameter to be chosen from the set of all stable rational function and determines the controller according to the Youla parametrization. In the analysis problem, Q is fixed and, as a result, so is M. For synthesis, we will need to find an appropriate Q which results in a controller providing satisfactory robustness properties. To do that, we adopt the following iteration scheme:

1. Set $i := 0$, and $R_o := I$.
2. Set $Q_i := arg\ \inf_{Q\ stable} \|R^{-1}(T_1 - T_2 Q T_3) R\|_{\mathcal{A}}$.
3. Set $R_i := arg\ \inf_{R \in \mathcal{R}} \|R^{-1}(T_1 - T_2 Q_i T_3) R\|_{\mathcal{A}}$.
4. Set $i := i + 1$. Go to step 2.

It is clear that this iteration converges, and furthermore the infimum values obtained in the consecutive application of steps 2 and 3 will be monotonically decreasing. It is also clear that the iteration procedure can be terminated at step 3 whenever a desireable robustness level is achieved as indicated by the value of the infimum at that step.

It remains to discuss the two optimization problems used in the iteration procedure above. The optimization problem in step 2 is a standard ℓ^1 optimization problem. This problem has been disussed in [6,8,9] and software packages for its solution exist and involve only linear programming. The second optimization problem, that appearing in step 3, can also be solved. Its solution is a direct application of the following lemma:

Lemma 1 *Let $M = T_1 - T_2QT_3$ with T_1, T_2, T_3, and Q stable. Let $|M|$ be as defined above. If $|M|$ is irreducible, then*

$$\inf_{R\in\mathcal{R}} \|R^{-1}MR\|_{\mathcal{A}} = \|\bar{R}^{-1}M\bar{R}\|_{\mathcal{A}},$$

where $\bar{R} := diag(\bar{r}_1,\ldots,\bar{r}_n)$, with $(\bar{r}_1,\ldots,\bar{r}_n)^T$ being the eigenvector corresponding to $\rho(|M|)$ which aside from being the spectral radius of $|M|$ will be an eigenvalue of $|M|$.

This lemma follows from the Perron-Frobenius theory for nonnegative matrices, and the proof will be omitted here. From this lemma, all that is needed to solve the optimization problem in step 3 will be to compute an eigenvector corresponding to the eigenvalue with the maximum modulus. Because we are dealing with nonnegative square matrices this eigenvalue turns out to be real and hence is itself equal to the spectral radius. Both the spectral radius and the eigenvalue corresponding to it can be computed very easily using power methods, another consequence of Perron Frobenius theory for nonnegative matrices. Finally, if $|M|$ were not in fact irreducible, it can be made so by replacing every zero entry with an $\epsilon > 0$. Since the spectral radius is a continuous function of the matrix entries, it follows that the solution of this modified problem will approach that of the original reducible one as ϵ approaches zero. Thus, the irreducibility assumption on $|M|$ is not a serious one, and the case when $|M|$ is reducible can be handled almost with the same ease as that when $|M|$ is irreducible.

We next look at a numerical example demonstrating the iteration scheme above.

5 Example 1

Consider the following plant family formed by adding weighted multiplicative perturbation to a

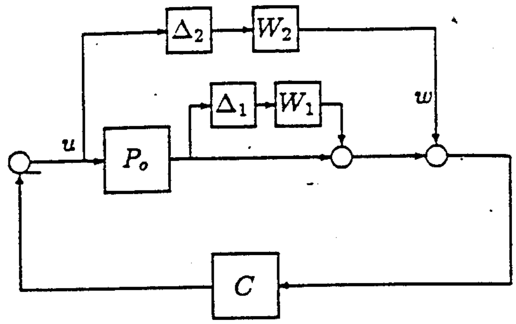

Figure 2: System considered in Example 1 after adding ficticious perturbation block for performance

nominal linear shift-invariant plant, P_o,

$$\Pi := \{P = (I + W_1\Delta_1)P_o \ : \ \Delta \in \boldsymbol{\Delta}\},$$

where $P_o = 3\frac{(\lambda+0.4)(\lambda-1.8)}{(\lambda-0.8)(\lambda+2)}$[1]. We will choose W_1 to be a high-pass FIR filter to reflect the fact that plant uncertainty is most common at high frequencies. MATLAB produced the following filter:

$$\begin{aligned} W_1 \ := \ & .0052\lambda^{10} - .008\lambda^9 - .0134\lambda^8 + .1057\lambda^7 \\ & -.2405\lambda^6 + .3072\lambda^5 - .2405\lambda^4 + .1057\lambda^3 \\ & -.0134\lambda^2 - .008\lambda + .0052 \end{aligned}$$

The uncertain system is subject to low frequency disturbance at the output. This disturbance is modelled as the output of a low-pass FIR filter, W_2. MATLAB was used to obtain the following filter:

$$\begin{aligned} W_2 := & -.0033\lambda^9 - .0162\lambda^8 + .1555\lambda^6 + .3641\lambda^5 \\ & +.3641\lambda^4 + .1555\lambda^3 - .0162\lambda - .0033 \end{aligned}$$

Our first objective is to achieve stability in the presence of uncertainty, i.e. we require the closed-loop system to be ℓ^∞-stable for all $P \in \Pi$. Our second objective, is to make the norm of the system from the disturbance input to to u less than one. This has the the effect of making the *magnitude* gain from the disturbance input to the control input less than one. This must of course be done in a worst case sense since we are dealing with a plant family, rather than a single plant. This problem is of practical importance when the control input magnitude is not to exceed certain rated values. We will use the iteration scheme discussed after transforming the problem to a

[1] In this paper, λ is equal to z^{-1}, where z is the familiar z-transform variable

stability robustness one with structured uncertainty. Fig. 2 shows the resulting stability robustness problem. It is straight forward to compute M which turns out to be:

$$M = \begin{pmatrix} -P_oC(I+P_oC)^{-1}W_1 & -P_oC(I+P_oC)^{-1}W_2 \\ -C(I+P_oC)^{-1}W_1 & -C(I+P_oC)^{-1}W_2 \end{pmatrix}$$

We now perform the iteration procedure. For convenience, we define $M(Q) = T_1 - T_2QT_3$.

- We set $R_o = I$. $\inf_Q \|M(Q)\|_{\mathcal{A}} = 1.538364$.
  Set $Q_o := arg \inf_Q \|M(Q)\|_{\mathcal{A}}$ [2].
- $\rho(|M(Q_o)|) = \inf_{R\in\mathcal{R}} \|R^{-1}M(Q_o)R\|_{\mathcal{A}} = 0.70956$.
  Set $R_1 := arg \inf_{R\in\mathcal{R}} \|R^{-1}M(Q_o)R\|_{\mathcal{A}}$.
- $\min_Q \|R_1^{-1}M(Q)R_1\|_{\mathcal{A}} = 0.703424$.
  Set $Q_1 := arg \inf_Q \|R_1^{-1}M(Q)R_1\|_{\mathcal{A}}$.
- $\rho(|M(Q_1)|) = \inf_{R\in\mathcal{R}} \|R^{-1}M(Q_1)R\|_{\mathcal{A}} = 0.681358$.
  Set $R_2 := arg \inf_{R\in\mathcal{R}} \|R^{-1}M(Q_1)R\|_{\mathcal{A}}$.
- $\min_Q \|R_2^{-1}M(Q)R_2\|_{\mathcal{A}} = 0.681184$.
  Set $Q_2 := arg \inf_Q \|R_2^{-1}M(Q)R_2\|_{\mathcal{A}}$.
- $\rho(|M(Q_2)|) = \inf_{R\in\mathcal{R}} \|R^{-1}M(Q_2)R\|_{\mathcal{A}} = 0.677072$.
  Set $R_3 := arg \inf_{R\in\mathcal{R}} \|R^{-1}M(Q_2)R\|_{\mathcal{A}}$.
- $\min_Q \|R_3^{-1}M(Q)R_3\|_{\mathcal{A}} = 0.677072$.
  Set $Q_3 := arg \inf_Q \|R_3^{-1}M(Q)R_3\|_{\mathcal{A}}$.
- $\rho(|M(Q_3)|) = \inf_{R\in\mathcal{R}} \|R^{-1}M(Q_3)R\|_{\mathcal{A}} = 0.677072$.

When starting points other than $R_o = I$ were chosen for the iteration the spectral radius to which the procedure converged did not change considerably from the one obtained here. Table 1 shows a few of these values for various starting points.

Needless to say, for the actual design we would use the Q parameter giving the smallest of these spectral radii. This would be the one obtained with $R_o = diag(10,1)$ as a starting point. It should be mentioned here that even though R has two parameters, the actual optimization problem $\inf_{R\in\mathcal{R}} \|R^{-1}MR\|_{\mathcal{A}}$ is a one dimensional one. Thus, only the ratio of the elements on the diagonal of R that affects the value of this infimum.

[2]The ℓ^1 optimization problems in this iteration example were solved by minimizing over all Q giving closed loop transfer function polynomial of order 20 or less.

Starting point R_o	Spectral radius to which iter. converged
$diag(1,1)$	0.677072
$diag(1,10)$	0.677072
$diag(1,100)$	0.665026
$diag(10,1)$	0.647499
$diag(100,1)$	0.648200

Table 1: Results of iteration for several R_o

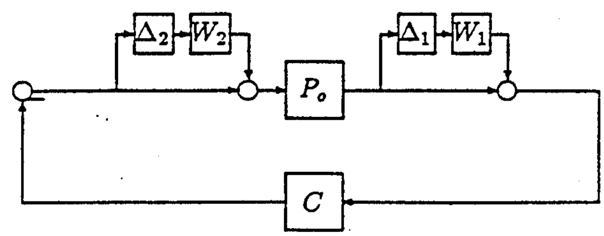

Figure 3: System for Example 2

6 Example 2

Given the nominal plant:

$$P_o = \frac{\lambda(\lambda-0.1)}{(\lambda-0.5)(\lambda-2)}$$

Suppose this nominal plant is subject to high frequency input and output uncertainy. This could be due to unmodelled sensor and actuator dynamics. This uncertainty is modelled by perturbation blocks Δ_1 and Δ_2 followed by high pass FIR filters W_1 and W_2 where

$$\begin{aligned} W_1 \;:=\; & -0.0037\lambda^8 - 0.007\lambda^7 + 0.0817\lambda^6 \\ & -0.2228\lambda^5 + 0.3\lambda^4 - 0.2228\lambda^3 \\ & +0.0817\lambda^2 - 0.007\lambda - 0.0037 \end{aligned}$$

and

$$\begin{aligned} W_2 \;:=\; & -0.0127\lambda^9 + .0248\lambda^8 + 0.0638\lambda^7 \\ & -0.2761\lambda^6 + 0.4\lambda^5 - 0.2761\lambda^4 \\ & +0.0638\lambda^3 + 0.0248\lambda^2 - 0.0127\lambda \end{aligned}$$

We are interested in maintaining system stability in the presence of the input and output perturbations. It can be easily seen that

$$M = \begin{pmatrix} -W_1P_oC(I+P_oC)^{-1} & W_2P_o(I+P_oC)^{-1} \\ -W_1C(I+P_oC)^{-1} & -W_2P_oC(I+P_oC)^{-1} \end{pmatrix}$$

We now apply the iteration scheme starting with $R_o = I$.

- $\inf_{Q\ stable} \|R_o^{-1}MR_o\|_{\mathcal{A}} = 1.0021.$
  Let $M_1 :=$ optimal M.

- $\inf_{R\in\mathcal{R}} \|R^{-1}M_1R\|_{\mathcal{A}} = 0.0332.$
  Let $R_1 :=$ optimal R.

- $\inf_{Q\ stable} \|R_1^{-1}MR_1\|_{\mathcal{A}} = 0.0330.$
  Let $M_2 :=$ optimal M.

- $\inf_{R\in\mathcal{R}} \|R^{-1}M_2R\|_{\mathcal{A}} = 0.0126.$

By lumping Δ_1 and Δ_2 together to form one multivariable Δ, i.e. by ignoring the structure of the perturbation, and obtaining a controller which is optimally robust for this Δ, one can only conclude that stability is maintained whenever $\|\Delta_i\| \leq \frac{1}{1.0021} = 0.997$. By applying the present analysis results on structured perturbations to the system with the controller obtained above, one sees that stability will in fact be maintained as long as $\|\Delta_i\| \leq 30.12$. Finally, if we use the controller corresponding to the last iteration step above, stability will be maintained whenever $\|\Delta_i\| \leq 79.4$. This demonstrates clearly the advantages of this robust synthesis scheme.

Acknowledgments

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Robust Disturbance Rejection in ℓ^1 Optimal Control Systems *

Mustafa Khammash and J.B. Pearson
Department of Electrical and Computer Engineering
Rice University, Houston, TX 77251-1892

Abstract

Given a class of plants formed by perturbing a nominal discrete-time linear shift-invariant plant with norm bounded unstructured perturbation, the problem of finding a single compensator that will stabilize all plants in this class and at the same time minimize the worst case norm of the sensitivity function is solved.

1 Introduction

When modeling physical systems as linear plants for the purpose of designing feedback controllers that make the closed loop system achieve certain specifications, one cannot escape the modelling uncertainties that are inherent in such a process. Even if the underlying physical system could be modelled exactly at one time, parameter variations that could appear for any one of many reasons eventually take their toll on the system and render the model inaccurate. For this reason, a controller that achieves good performance when controlling the model, might not perform so well when used to control the actual plant and could even make the system unstable. Therefore, robustness of the control system to variations in the plant are of great practical importance. Stability robustness can be achieved if the controller can be made to stabilize a whole family of plants. Performance robustness, on the other hand, can be achieved if in addition the controller can be chosen so as to give "good" performance for each one of the members of the plant class. Stability robustness is therefore required for performance robustness. In this respect, recent work by M. Dahleh and Y. Ohta [1] provides necessary and sufficient conditions for BIBO stability robustness. The plant perturbations considered in [1] take the form of multiplicative or additive perturbations with a bounded norm. In addition, the perturbations are allowed to be time-varying or nonlinear.

This paper considers performance robustness when the performance criterion is ℓ^∞ disturbance rejection. Good performance, in this case, translates into small norms for certain loop functions, e.g. the sensitivity function. Accordingly, in the case of sensitivity, robust performance can be achieved if the norm of the sensitivity function can be made small for all perturbed plants, an objective that can be achieved by minimizing, with the proper choice of a robustly stabilizing controller, the *worst* case norm of this function.

2 Problem Statement

Let P_o be a given nominal discrete-time plant. P_o is assumed to be linear, shift-invariant, and strictly causal with q inputs and p outputs.

Denote by $S(P_o)$ the set of all linear shift-invariant discrete-time controllers with the appropriate dimension that stabilize P_o. We now define a family of plants formed by adding weighted multiplicative perturbations to the nominal plant. Let

$$\Pi := \{P : \; P = P_o(I + W_1\Delta)\}$$

where $W_1 \in \mathcal{L}_{TI}^{q\times q}$ and $\Delta : \ell_q^\infty \to \ell_q^\infty$ is causal with $\|\Delta\| := \sup_{x\neq 0} \frac{\|\Delta x\|_\infty}{\|x\|_\infty} \leq 1$. So Δ is allowed to be time-varying or nonlinear. We also define

$$\Psi := \{C \in S(P_o) : \; C \text{ stabilizes all } P \in \Pi\}.$$

When performance is measured by the norm of the weighted sensitivity function, the problem of achieving robust performance and stability can now be stated as follows:

$$\inf_{C\in\Psi} \sup_{P\in\Pi} \| \left((I + CP)^{-1}W_2\right) \| =: \gamma_{opt}$$

where $W_2 \in \mathcal{L}_{TI}^{q\times q}$.

It is therefore desired to compute γ_{opt} and to find a controller $C \in \Psi$ that will make the quantity $\sup_{P\in\Pi} \|(I + CP)^{-1}W_2\|$ arbitrarily close to γ_{opt}.

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3 Problem Solution

Theorem 3.1, to be presented next, is essentially the key to solving the problem posed earlier. Together with Corollary 3.2 and Theorem 3.3, it forms the main result in this paper. The proof of Theorem 3.1 is rather involved and will not be presented here. See [4] for a complete proof. We will instead demonstrate how these results can be utilized to solve the stated problem. In what follows, if $R \in \mathcal{L}_{TI}^{q\times q}$ then R_i will denote the ith row of the transfer function matrix of R.

Theorem 3.1. *Let T and S both be in $\mathcal{L}_{TI}^{q\times q}$ with T satisfying $\|T\|_{\mathcal{A}} < 1$. Then*

$$\sup_{\substack{\Delta \text{causal} \\ \|\Delta\|\leq 1}} \|(I+T\Delta)^{-1}S\| = \max_{1\leq i\leq q} \frac{\|S_i\|_{\mathcal{A}}}{1-\|T_i\|_{\mathcal{A}}}.$$

Corollary 3.2. *Let $C \in S(P_o)$ such that $\|(I+CP_o)^{-1}CP_oW_1\|_{\mathcal{A}} < 1$. Then*

$$\sup_{P\in\Pi}\|(I+CP)^{-1}W_2\| =$$

$$\max_{1\leq i\leq q} \frac{\|\left((I+CP_o)^{-1}W_2\right)_i\|_{\mathcal{A}}}{1-\|\left((I+CP_o)^{-1}CP_oW_1\right)_i\|_{\mathcal{A}}}$$

Proof : Define $T := (I+CP_o)^{-1}CP_oW_1$ and $S := (I+CP_o)^{-1}W_2$. $(I+CP)^{-1}W_2$ can be expanded as follows:

$$(I+CP)^{-1}W_2 = (I+T\Delta)^{-1}S.$$

Now applying Theorem 3.1 gives the desired result. ∎

The next theorem is a consequence of Corollary 3.2 and the result of Dahleh and Ohta [1] concerning conditions for stability robustness.

Theorem 3.3. *Let $C \in S(P_o)$, and let $\gamma > 0$. Then*

$$C \text{ stabilizes every } P \in \Pi \quad \text{and}$$
$$\sup_{P\in\Pi}\|(I+CP)^{-1}W_2\| < \gamma$$
$$\text{if and only if}$$
$$\left\|\left[(I+CP_o)^{-1}W_2 \quad \gamma(I+CP_o)^{-1}CP_oW_1\right]\right\|_{\mathcal{A}} < \gamma.$$

Proof : See [4].

Theorem 3.3 suggests a way to minimize the quantity $\sup_{P\in\Pi}\|(I+CP)^{-1}W_2\|$ subject to robust stability, by which γ_{opt} can be approached arbitrarily closely and a controller that achieves this can be found. Provided robust stability *can* be achieved, it is easy to see how iteration on the parameter γ and solving an $\mathcal{A}$-norm minimization problem at each step will achieve the desired minimization. See [4] for more details on the iteration scheme and [3,5,6] for the techniques of solving the $\mathcal{A}$-norm minimization problems.

4 Conclusion

In this paper, it has been shown how stability robustness and performance robustness can be incorporated together in one design procedure when the performance is measured by the norm of the sensitivity function. An expression for the worst case norm of this function has been provided when norm bounded perturbations are present. Such an expression provides an effective way of combining both robust stability and performance in one, easy to compute, measure. Furthermore, this expression can be minimized subject to robust stability constraints to provide a controller with optimal robustness properties. Finally, it should be mentioned that even though the perturbations considered here were multiplicative perturbations, the situation is almost identical when additive perturbations are assumed.

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ROBUSTNESS IN THE PRESENCE OF STRUCTURED UNCERTAINTY *

M. Khammash† and J.B. Pearson‡

ABSTRACT

Recent developments in the robustness of systems with structured norm-bounded perturbations are presented. The stability and performance robustness of linear time-invariant systems with $\ell^\infty/\mathcal{L}^\infty$ norm-bounded structured uncertainty is discussed. Moreover, new results on the robustness of time-varying systems including necessary and sufficient conditions for stability robustness are discussed. It is shown that for both time-varying as well as time-invariant systems nonconservative robustness conditions can be obtained in terms of certain spectral radii of nonnegative matrices obtained from the nominal system. The robustness conditions are shown to be computable even for a large number of uncertainty blocks.

Key Words: Robustness, Structured Uncertainty, ℓ^1 systems

1. INTRODUCTION

Robustness in the face of structured uncertainty is an important objective of control. As models of physical systems rarely correspond exactly to the true systems they are supposed to model, it is necessary to account for the resulting uncertainty both in the design and analysis procedures. Previous work on the robustness problem using the ℓ^∞ signal norm has been done by Dahleh and Ohta [1] who solve the stability robustness problem in the case of *unstructured* perturbations and for time-invariant systems. For time-varying systems, Shamma and Dahleh [2] provide necessary conditions for robust stability for systems with unstructured perturbations. This paper discusses the stability and performance robustness of systems in the presence of *structured* uncertainty. Each uncertainty block has an induced ℓ^∞ norm which is bounded. For such uncertainty, and when the nominal system composed of the nominal plant and controller are linear time-invariant, necessary and sufficient conditions for robust stability are presented. These conditions are stated in terms of the spectral radius of a certain nonnegative matrix obtained from the nominal system and hence can be computed for a very large number of uncertainty blocks. In addition, the relationship between stability and performance robustness is

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†Elec. Eng. and Comp. Eng., Iowa State Univ., Ames, Iowa
‡Elec.& Comp. Eng., Rice Univ., Houston, Texas

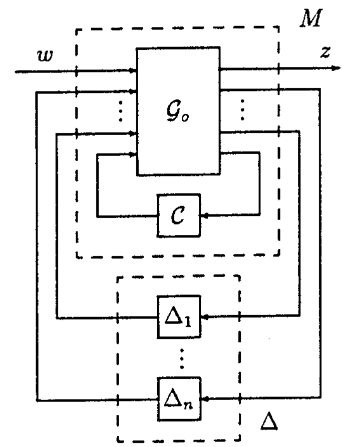


Fig. 1. System with Structured Uncertainty.

highlighted. By showing a certain equivalence between stability and performance robustness, the problem of achieving robust performance in the presence of structured uncertainty can be reduced to a robust stability problem of another system. This allows the treatment of stability and performance robustness in the same framework.

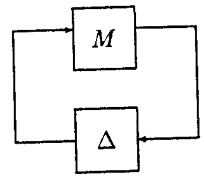
In many situations, the nominal system composed of the nominal plant and the stabilizing controller may be time-varying. This is the case for example when dealing with adaptive control systems or sampled-data systems. Time-varying nominal systems can also arise when time-varying weights are used in shaping certain signals or in modelling uncertainty. When the nominal system is time-varying, necessary and sufficient conditions for the robustness of time-varying systems are provided. These conditions are expressible in terms of the spectral radius of a parametrized family of matrices obtained from the kernel representation of the nominal system.

This paper is organized as follows. In section 2 the robustness problem in the presence of structured perturbations is set up. In section 3 the robustness of time-invariant systems is discussed, and necessary and sufficient conditions are provided for both stability and performance robustness. In section 4 the robustness of time-varying systems is addressed, and necessary and sufficient conditions are provided for stability robustness. Finally, section 5 contains some concluding remarks.

2. PROBLEM SETUP

The standard setup for a general robustness problem appears in Fig. 1. In the figure, $\mathcal{G}_o$ is a nominal linear plant. Since all the results in this paper hold for continuous and discrete-time systems with the obvious modifications, $\mathcal{G}_o$ may be continuous-time or discrete-time. $\mathcal{C}$ is a linear controller stabilizing $\mathcal{G}_o$. For the analysis problem, $\mathcal{C}$ is assumed given and fixed. The uncertainty is modelled with perturbation blocks $\Delta_1, \ldots, \Delta_n$. Each perturbation Δ_i belongs to the following class of admissible perturbations:

$$\underline{\Delta} = \{\Delta \ : \ \Delta \text{ is causal, and } \|\Delta\| \ := \sup_{u \neq 0} \frac{\|\Delta u\|_\infty}{\|u\|_\infty} \leq 1\}, \tag{1}$$



$\Delta \in \mathcal{D}(n)$

Fig. 2. Stability robustness problem

where the norm used is the $\mathcal{L}^\infty$ norm (or ℓ^∞ norm for discrete-time systems). The perturbations may therefore be nonlinear or time-varying. w is an exogenous bounded disturbance, and z is a regulated output. The n perturbation blocks can be lumped into one perturbation block with a diagonal structure. Hence we can view the class of admissible perturbations as the class of all $\Delta \in \mathcal{D}(n)$ where

$$\mathcal{D}(n) := \{diag(\Delta_1, \cdots, \Delta_n) \; : \; \Delta_i \in \underline{\Delta}\}. \tag{2}$$

Similarly, $\mathcal{G}_o$ and $\mathcal{C}$ can be lumped into one system M. M is therefore, linear, causal, and stable. Any weighting on any of the perturbations can be lumped into M.

The system in the figure is said to robustly stable if it is $\mathcal{L}^\infty$-stable for all admissible perturbations, i.e. for all $\Delta \in \mathcal{D}(n)$. It is said to achieve robust performance if it achieves robust stability and satisfies:

$$\|\mathcal{T}_{zw}\| < 1 \qquad \forall \; \Delta \in \mathcal{D}(n), \tag{3}$$

where $\mathcal{T}_{zw}$ is the map from w to z, and the norm used is the induced operator norm.

In the next two sections, we provide necessary conditions for robustness when M is time-invariant and when M is time-varying. We begin with the former.

3. ROBUSTNESS OF TIME-INVARIANT SYSTEMS

We start the discussion of the time-invariant case by first addressing conditions for robust stability alone. Following the treatment of robust stability, we address the robust performance problem.

3.1 Stability Robustness

Consider the system in Fig. 2. From the figure, M has n inputs and n outputs corresponding to the inputs and outputs of the perturbations. Each M_{ij} has induced norm which we refer to as the $\mathcal{A}$ norm. It can be computed arbitrarily accurately since $\|M_{ij}\|_{\mathcal{A}} = |D_{ij}| + \sum_{k=0}^{\infty} |C_i A^k B_j|$ in the discrete time case, and $\|M_{ij}\|_{\mathcal{A}} = |D_{ij}| + \int_0^\infty |C_i e^{At} B_j| dt$ in the continuous-time case, where A, B_i, C_j, D_{ij} are the constant matrices in the state-space description of M_{ij}. We can therefore define the following matrix:

$$\widehat{M} = \begin{bmatrix} \|M_{11}\|_{\mathcal{A}} & \cdots & \|M_{1n}\|_{\mathcal{A}} \\ \vdots & & \vdots \\ \|M_{n1}\|_{\mathcal{A}} & \cdots & \|M_{nn}\|_{\mathcal{A}} \end{bmatrix} \tag{4}$$

As the next theorem shows, it turns out that $\widehat{M}$ plays a fundamental role in the robustness of the given system. We now state the main theorem establishing non-conservative conditions for robustness:

Theorem 1 *The system in Fig. 2 achieves robust stability if and only if any one of the following conditions holds:*

1. $\rho(\widehat{M}) < 1$, *where* $\rho(.)$ *denotes the spectral radius.*
2. $\inf_{R \in \mathcal{R}} \|R^{-1}MR\|_{\mathcal{A}} < 1$ *where* $\mathcal{R} := \{diag(r_1, \ldots, r_n) : r_i > 0\}$.

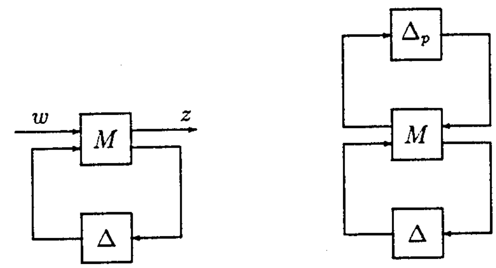
The proof of this theorem can be found in [3] for the discrete-time case and in [4] for the continuous-time. Since this theorem reduces the robustness analysis problem to that of computing the spectral radius of a nonnegative matrix. The theory for nonnegative matrices (see e.g. [5]) provides power algorithms for fast computation of the spectral radius of a nonnegative matrix. As a result, the robustness conditions can be computed exactly and efficiently which are especially suited for systems with a large number of uncertainty blocks. The second condition is useful for the synthesis of robust controllers since it turns out from the theory of nonnegative matrices that under very mild conditions on $\widehat{M}$ the infimum is in fact achieved by a certain matrix R which is formed by writing the positive eigenvector corresponding to $\rho(\widehat{M})$, itself an eigenvalue of $\widehat{M}$, along the diagonal of R and setting all other entries to zero. In addition, if $\widehat{M}$ does not satisfy the required conditions it can be made to do so by perturbing it slightly while keeping its spectral radius as close as desired to the original one. So a procedure for finding a controller that makes the spectral radius small can be devised similar to the D-K iteration used in the μ synthesis (see e.g. [6]). The differences between this case and the μ synthesis is that the scaling matrix R in this case is constant and not frequency dependent and thus is easier to compute. Moreover, unlike the μ, condition *2.* in the theorem above remains necessary even for $n > 3$. These differences are attributed mainly to the difference in norms used as well as the class of allowed perturbations.

3.2 Robust Performance

Thus far we have discussed only robust stability. It turns out that in the time-invariant M case, robust performance can be treated in the same framework as robust stability thanks to a special equivalence relationship between the two. The equivalence is the subject of the next main theorem. But first consider the two systems in Fig. 3. SYSTEM I is the one for which we seek robust performance. SYSTEM II is formed from SYSTEM I by feeding the output z back to the input w through a perturbation block Δ_P. Robust stability of SYSTEM II is closely related to robust performance of SYSTEM I. This is what the following equivalence theorem states:

Theorem 2 *With SYSTEM I and SYSTEM II as in Fig. 3, SYSTEM I achieves robust performance if and only if SYSTEM II achieves robust stability.*

As mentioned earlier the robust stability of SYSTEM II is equivalent to stability for all $diag(\Delta_P, \Delta) \in \mathcal{D}(n+1)$, and can be tested using the spectral radius test in Theorem 1. The proof of Theorem 2 can be found in [3]. Even though one direction of the proof is fairly obvious and follows directly from the Small Gain theorem, the proof of the other direction requires some results on the stability robustness of time-varying systems.



Perturbation class $\mathcal{D}(n)$ SYSTEM I   Perturbation class $\mathcal{D}(n+1)$ SYSTEM II

Fig. 3. Stability robustness vs. performance robustness

4. ROBUSTNESS OF TIME-VARYING SYSTEMS

We now discuss the general case when M is time-varying. Of special interest is the case when M is periodically time-varying. Such M arise when dealing with sampled-data systems. For time-varying systems various properties of the norm which hold for time-invariant systems cease to hold. In particular if $(\,M_1 \quad M_2\,)$ is a time-varying system then unlike the time-invariant case, $\|\,(\,M_1 \quad M_2\,)\,\|$ is *not* equal to $\|M_1\|_{\mathcal{A}} + \|M_2\|_{\mathcal{A}}$. Many of the subtle differences in the robustness conditions between time-varying and time-invariant systems are attributed to this fact. Another property which time-varying systems do not possess is that of commuting with the shift operator. We define the shift operator for time-varying systems as follows:

$$\begin{aligned} S_T \quad &: \quad \mathcal{L}_e^\infty \mapsto \mathcal{L}_e^\infty \quad \text{such that} \\ (S_T u)(t) \quad &= \quad \begin{cases} u(t-T) & \text{whenever } t \geq T \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{5}$$

For the robustness of time-varying systems the operator $S_{-T} M S_T$ plays an important role. Before we can state necessary and sufficient conditions for stability robustness of time-varying systems we need a representation for the time-varying operator M_{ij}. Since M_{ij} is a linear, causal, and stable map, it has a certain kernel representation, say $M_{ij}(t,\tau)$ for $0 \leq t, \tau < \infty$, so that for any $u \in \mathcal{L}^\infty$

$$(M_{ij}u)(t) = \int_0^\infty M_{ij}(t,\tau)u(\tau)d\tau \tag{6}$$

where $M_{ij}(t,\tau) = \hat{M}_{ij}(t,\tau) + \sum_{k=0}^\infty \hat{m}_{ij}^k(t)\delta(t - t_k - \tau)$ (see [7] for more details). Because M_{ij} is $\mathcal{L}^\infty$-stable it holds that

$$ess \sup_t \int_0^\infty |\hat{M}_{ij}(t,\tau)|d\tau + \sum_{k=0}^\infty |\hat{m}_{ij}^k(t)| < \infty.$$

We can suppress the dependence of M_{ij} on τ by writing $M_{ij}(t)$ by which we mean the function $M_{ij}(t,\cdot)$. This belongs to the algebra $\mathcal{A}$ (see [7]). In this case, $\|M_{ij}(t)\|_{\mathcal{A}} = \int_0^t |\hat{M}_{ij}(t,\tau)|d\tau + \sum_{k=0}^\infty |\hat{m}_{ij}^k(t)|$. It can be verified that

$$\|M_{ij}\| = \sup_t \|M_{ij}(t)\|_{\mathcal{A}}. \tag{7}$$

We are now in a position to state the generalization of Theorem 1 to time-varying systems.

Theorem 3 *For the system in Fig. 2 and with M a stable and causal time-varying operator robust stability is achieved if and only if any of the following two conditions are met:*

1. *For some $T > 0$*

$$\sup_{t_i \geq 0} \rho \left(\begin{bmatrix} \|(S_{-T}M_{11}S_T)(t_1)\|_{\mathcal{A}} & \cdots & \|(S_{-T}M_{1n}S_T)(t_1)\|_{\mathcal{A}} \\ \vdots & & \vdots \\ \|(S_{-T}M_{n1}S_T)(t_n)\|_{\mathcal{A}} & \cdots & \|(S_{-T}M_{nn}S_T)(t_n)\|_{\mathcal{A}} \end{bmatrix} \right) < 1.$$

2. *For some $T > 0$,* $\quad \inf_{R \in \mathcal{R}} \|R^{-1}S_{-T}MS_TR\| < 1.$

This theorem appears in [4] where the proof can be found. For periodically time-varying systems various simplifications take place in the statement of Theorem 3. In particular, for sampled-data systems state-space formulae can be obtained for the quantities appearing in the theorem statement and the norms can be computed arbitrarily accurately. Furthermore, the supremum in item *1* of theorem 3 can be taken over a compact set in the case of sampled-data systems. More details about these computations appear in [4].

5. CONCLUSIONS

In this paper, computable necessary and sufficient conditions for the robustness of time-invariant systems in the presence of structured uncertainty were presented. It was shown that performance robustness can be handled in the same framework as stability robustness. Finally, necessary and sufficient conditions in terms of the spectral radius were given for the robustness of time-varying systems. For time-varying systems it can be shown that the relationship that exists between stability robustness and performance robustness in the time-invariant case ceases to hold for time-varying systems. More work needs to be done in this direction.

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Uniform Stability and Performance in H_∞*

A. P. Kishore J. B. Pearson, Jr.

Department of Electrical and Computer Engineering
Rice University, Houston, TX 77251-1892, U.S.A.

ABSTRACT

We consider robust stability and performance analysis problems for continuous-time single-input single-output plants in the H_∞ setting. For a multiplicative uncertainty model, we show that well-known conditions for stability and performance are not necessary conditions. We show there is no equivalence between the stability and performance problems. We argue that stability of M-Δ configuration is not always equivalent to robust stability. We consider uniform stability and uniform performance, and examine their relationship with each other.

NOTATION

$\mathbb{C}_{+e}$	$\{s \in \mathbb{C} : \text{Re}\, s \geq 0\} \cup \{\infty\}$
$\Omega\,(\bar{\Omega})$	open (closed) right half plane
$\mathbf{H}_\infty$	space of bounded holomorphic functions on Ω
$\mathbf{RH}_\infty$	rational functions in H_∞ with real coefficients
$\bar{\mathcal{A}}$	space of bounded holomorphic functions on $\bar{\Omega}$
$\mathbf{\Delta}$	$\{\Delta \in \bar{\mathcal{A}} : \|\Delta\|_\infty \leq 1\}$
$\mathbf{\Delta}^o$	$\{\Delta \in \bar{\mathcal{A}} : \|\Delta\|_\infty < 1\}$

1. INTRODUCTION

We consider the robust stability and performance problems for single-input single-output continuous-time plants with uncertainty in the H_∞ setting. Consider the feedback system in Figure 1, where r is the reference input, y is the controlled output, and d is the disturbance. In the figure, C is the controller, and $\tilde{P}$ is the plant, both represented minimally. We say that the system in Figure 1 is *well-posed* if for any locally square-integrable r, d there is a unique pair x_1, x_2 with x_1, x_2 locally square-integrable. We say that the system in Figure 1 is *internally stable* or that C *stabilizes* $\tilde{P}$ if it is well-posed and if the four transfer functions from (r, d) to (x_1, x_2) are stable (i.e. in H_∞). We say that C *achieves performance with respect to property* p *for* $\tilde{P}$ if it stabilizes $\tilde{P}$ and if p holds. If it is understood what p is, we simply say that C achieves performance for $\tilde{P}$. In this paper, $\|W_1 T_{x_1 r}\|_\infty < 1$ is the performance property where $W_1 \in RH_\infty$ is fixed and $T_{x_1 r}$ is the map from r to x_1.

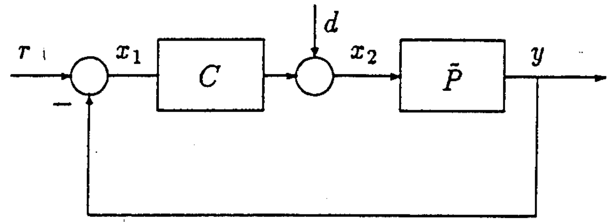

Figure 1: Feedback configuration

Mathematical models rarely describe the plant exactly. That is, $\tilde{P}$ is not known precisely, but can only be placed within a given *uncertainty set* Π. In such a case, we are interested in designing a single controller which stabilizes or achieves performance for *every* $\tilde{P} \in \Pi$. We have the following definitions.

Definition 1.1. The controller C *robustly stabilizes* Π if C stabilizes every $\tilde{P} \in \Pi$.

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Definition 1.2. The controller C achieves *robust performance* for Π if C achieves performance for every $\tilde{P} \in \Pi$.

Then the robust stability (performance) problem amounts to determining if a given controller C achieves robust stability (performance) for a given uncertainty set. A typical uncertainty set is $\{\tilde{P} = P + \Delta : \Delta \in H_\infty, \|\Delta\|_\infty \leq 1\}$, where P is called the *nominal plant* and is a distinguished member of the set, and Δ is called the perturbation on P. We then have notions of nominal stability and nominal performance whose meaning is obvious. Several uncertainty sets parametrized by *stable* perturbations on a nominal plant such as additive and multiplicative perturbations [1] have been considered in the literature. For the single-input single-output systems, the uncertainty is *unstructured* if a scalar perturbation parametrizes the uncertainty set. Otherwise it is called structured. In this note, we consider both structured and unstructured uncertainty.

The reason for elaborating on the definitions at length is that we will show in Section 3 that a standard approach, which we call "M-Δ analysis", does not always solve the robust stability problem as stated above. There are necessary and sufficient conditions for both robust stability and robust performance [1], if a) the nominal plant and controller do not have poles and zeros on the imaginary axis, and b) if the uncertainty is unstructured. We show that if we relax *either* a) or b), the conditions are sufficient but not necessary. We consider robust stability in Section 2, a comparison of M-Δ analysis in Section 3, robust performance in Section 4, and their relationship in Section 5.

Consider a controller C which robustly stabilizes a class of plants Π. Then, given any $\tilde{P} \in \Pi$, the four transfer functions from (r, d) to (x_1, x_2) are stable, and hence have finite norms. But it is plausible that as $\tilde{P}$ varies over Π these norms can grow arbitrarily large. That is, as we show in Section 6, there may not be a *uniform* bound on the norms of these transfer functions as $\tilde{P}$ varies over Π. This is why we have robust performance problem, where we ensure that weighted norm of a transfer function remains bounded under perturbations. We show in Section 6 that even when we have robust performance, norms of some transfer functions can grow arbitrarily large. It is desirable that we design a controller C that achieves such a uniform norm bound on *all* transfer functions. With this motivation, we consider uniform stability (performance) problems and examine their relationship with each other and with robust stability (performance) problems. We compare using M-Δ analysis to solve these problems. A discussion on the results follows in the last section.

2. Robust Stability

Internal stability for the feedback configuration under consideration is equivalent to the stability of the three transfer functions

$$\frac{\tilde{P}}{1+\tilde{P}C}, \ \frac{C}{1+\tilde{P}C}, \text{ and } \frac{1}{1+\tilde{P}C}.$$

We consider unstructured uncertainty first. Define

$$\Pi_M := \{\tilde{P} = P(1+\Delta W_2) : \ \Delta \in \bar{A}, \|\Delta\|_\infty \leq 1\},$$

$$\Pi_M^\circ := \{\tilde{P} = P(1+\Delta W_2) : \ \Delta \in \bar{A}, \|\Delta\|_\infty < 1\},$$

where $W_2 \in RH_\infty$ is fixed. We assume the following throughout the paper.

(A1) The nominal plant and the controller are proper rational functions with real coefficients.

There are no assumptions on the location of poles or zeros of either the nominal plant or the controller. In particular, we do not assume they do not have poles on the imaginary axis. Assumption A1 is not crucial in the paper. All the results hold verbatim for distributed plants having finitely many poles of finite order in the closed right half plane. Define

$$T := \frac{PC}{1+PC}, \ S := \frac{1}{1+PC}.$$

Proposition 2.1. *Let C stabilize the nominal plant P. Then:*

(i) C robustly stabilizes Π_M° if and only if $\|W_2T\|_\infty \leq 1$.

(ii) $\|W_2T\|_\infty < 1$ implies C robustly stabilizes Π_M.

(iii) C robustly stabilizes Π_M does not imply $\|W_2T\|_\infty < 1$.

(iv) C robustly stabilizes Π_M implies $\|W_2T\|_\infty \leq 1$.

(v) In addition, assume that neither P nor C has poles on the imaginary axis. Then, C robustly stabilizes Π_M if and only if $\|W_2T\|_\infty < 1$.

Part (ii) follows from small gain theorem [5]. Parts (i), (iv) and (v) are easy to prove. We prove (iii) by exhibiting an example with $\|W_2T\|_\infty = 1$ where there is no destabilizing $\Delta \in \mathbf{\Delta}$.

Example 1. Let $P(s) := \frac{1}{s}$, $C(s) := 1$, and $W_2(s) := 1$.

Proof. For rational perturbations, a destabilizing Δ needs to satisfy $\Delta(0) = -1$ and $\Delta'(0) = -1$. Such a Δ doesn't belong to $\mathbf{\Delta}$. □

See [3] for the case of general perturbations in $\mathbf{\Delta}$.

In [4], it is shown that a necessary and sufficient robustness condition with a non-strict inequality exists for a class of stable-factor perturbations defined with a strict inequality. It is also shown that if the inequalities are switched, the condition becomes sufficient but not necessary. Our construction is similar to the construction of the counter-example there.

We now consider structured uncertainty. Define

$$\tilde{\Pi} := \{P\frac{1+\Delta_2W_2}{1+\Delta_1W_1} : \Delta_1, \Delta_2 \in \mathbf{\Delta}, \Delta_1 \not\equiv -W_1^{-1}\},$$

$$\tilde{\Pi}^o := \{P\frac{1+\Delta_2W_2}{1+\Delta_1W_1} : \Delta_1, \Delta_2 \in \mathbf{\Delta}^o, \Delta_1 \not\equiv -W_1^{-1}\},$$

where $W_1, W_2 \in RH_\infty$ are fixed.

Proposition 2.2. *Let C stabilize P. Then:*

(i) C robustly stabilizes $\tilde{\Pi}^o$ if and only if $\|\,|W_1S| + |W_2T|\,\|_\infty \leq 1$.

(ii) $\|\,|W_1S| + |W_2T|\,\|_\infty < 1$ is sufficient but not necessary for C to robustly stabilize $\tilde{\Pi}$.

(iii) C robustly stabilizes $\tilde{\Pi}$ implies $\|\,|W_1S| + |W_2T|\,\|_\infty \leq 1$.

Parts (i), (iii), and sufficiency in (ii) are easy to prove. The following example completes the proof of (ii). For details, see [3].

Example 2. Let $P = \frac{1}{s+1}, C = 1, W_1 = 1$, and $W_2 = \frac{1}{s+1}$.

Note that both the nominal plant and controller in the example are rational functions with no poles and zeros on the imaginary axis. This should be contrasted with Proposition 2.1(v).

3. M-Δ Analysis

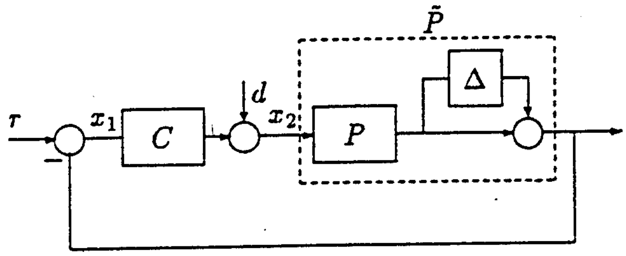
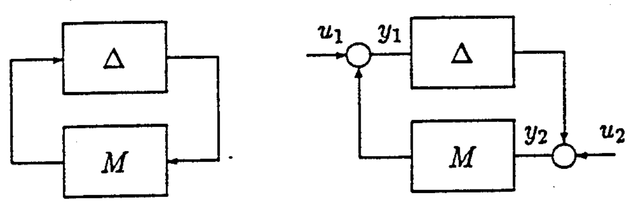


Figure 2: Multiplicative Perturbations

In the literature, there is one approach to solve the robust stability problem of Section 1, which has been generalized to deal with *structured uncertainty* allowing multiple uncertainties at several locations in the plant. This approach is to rearrange the given configuration, redrawn in Figure 2 with Δ in place, to match Figure 3(a). Here, M is the transfer function from the output of Δ to the input of Δ. Then the "stability of the closed loop system" in this figure is studied. To define this, two additional *fictitious* signals are introduced as in Figure 3(b). The M-Δ configuration is called stable if the four transfer functions from (u_1, u_2) to (y_1, y_2) are stable. This is shown to be equivalent to the invertibility of $(I - M\Delta)$ in H_∞ for every Δ in the unit ball. Then a necessary and sufficient condition for this, a version of the small gain theorem, is derived. For example, the following holds for unstructured uncertainty.



(a) (b)

Figure 3: General M-Δ structure

Lemma 3.1. *$(I - M\Delta)^{-1} \in H_\infty$ for all $\Delta \in RH_\infty$ with $\|\Delta\|_\infty \leq 1$ if and only if $\|M\|_\infty < 1$.*

We can interchange $\leq$ and $<$ in this lemma. Similarly, for structured uncertainty which gives rise to a block-diagonal Δ and a transfer function matrix M, we have the μ-test [1] to determine the stability of the M-Δ configuration. With the previous examples, it is clear that robust stability of the configuration in Figure 2 need not imply the stability of the M-Δ configuration. Notice we have shown this in Example 2 with a plant and controller that are rational and stable. That is, the M-Δ analysis may not always solve the robust stability problem stated in Section 1. Presumably, the equivalence between these two notions of stability depends on the norm, the set of Δ (e.g. open or closed unit ball), and other assumptions on the plant or controller (e.g. no poles of controller or plant on the imaginary axis), and the perturbation class itself (e.g. they are equivalent for additive perturbations, but not for multiplicative perturbations). This equivalence issue is an open problem.

4. Robust Performance

The performance criterion under consideration is that the H_∞ norm of the map from r to x_1 weighted by W_1 be strictly less than 1. Then *robust performance for Π_M (Π_M^o)* may be defined as

robust stability for Π_M (Π_M^o) and

$$\left\|\frac{W_1 S}{1+\Delta W_2 T}\right\|_\infty < 1, \quad \forall\, \Delta \in \boldsymbol{\Delta}\ (\boldsymbol{\Delta}^o).$$

We claim the following.

Proposition 4.1. *Let C stabilize the nominal plant P. Then:*

(i) Robust performance for $\Pi_M^o \iff \|\,|W_1 S| + |W_2 T|\,\|_\infty \leq 1$.

(ii) $\|\,|W_1 S| + |W_2 T|\,\|_\infty < 1$ implies robust performance for Π_M.

(iii) Robust performance for Π_M does not imply $\|\,|W_1 S| + |W_2 T|\,\|_\infty < 1$.

(iv) Robust performance for Π_M implies $\|\,|W_1 S| + |W_2 T|\,\|_\infty \leq 1$.

(v) In addition, assume that neither P nor C has poles on the imaginary axis. Then, robust performance for $\Pi_M \iff \|\,|W_1 S| + |W_2 T|\,\|_\infty < 1$.

As we can have robust stability with $\|W_2 T\|_\infty = 1$, it may seem that (iii) is immediate. However, it may happen that for any robustly stable system with $\|W_2 T\|_\infty = 1$, there is no W_1 such that a performance bound is achieved. Proof of (i), (ii), (iv) and (v) is simple. We establish (iii) by an example.

Example 3. Consider the Example 1 with $W_1(s) := \frac{0.49s}{1+s}$. For each ω and each $\Delta \in \boldsymbol{\Delta}$, we have

$$\left|\frac{W_1 S}{1+\Delta T}\right| \leq \frac{|W_1 S|}{1-|T|} = \frac{0.49\omega^2}{1+\omega^2-\sqrt{1+\omega^2}} \leq 0.98.$$

So, C achieves robust performance, but $\|\,|W_1 S| + |W_2 T|\,\|_\infty \geq 1$. □

5. Relation between Performance and Stability

We now examine if robust performance for Π_M is equivalent to robust stability for $\bar{\Pi}$. The equivalence breaks down rather miserably, but the two notions are not completely unrelated. We will compare the case of Π_M with that of Π_M^o. The main result of this section follows.

Proposition 5.1. *Let C stabilize P. Then:*

(i) Robust performance for $\Pi_M^o \iff C$ robustly stabilizes $\bar{\Pi}^o$.

(ii) Robust stability for $\bar{\Pi}$ does not imply nominal performance for Π_M.

(iii) Robust performance for Π_M implies robust stability for $\bar{\Pi}$.

Proof of (i) follows from Proposition 2.3(i) and Proposition 4.4(i). Again we prove (ii) by an example. For proof of (iii), see [3].

Example Consider Example 2 again. We have shown that C robustly stabilizes $\bar{\Pi}$. However, since $\|W_1 S\|_\infty = 1$, the closed loop system does not even have nominal performance. □

6. Uniform Stability and Uniform Performance

In Section 1 we discussed a notion of robust stability and performance. In this section we

consider another notion of stability and performance. Consider a controller C which robustly stabilizes a class of plants Π. Then, given any $\tilde{P} \in \Pi$, the three transfer functions

$$\frac{1}{1+\tilde{P}C}, \ \frac{C}{1+\tilde{P}C}, \text{ and } \frac{\tilde{P}}{1+\tilde{P}C}$$

are stable, and hence have finite norms. But it is conceivable that as $\tilde{P}$ varies over Π these norms can grow arbitrarily large. In other words, there may not be a *uniform* bound on the norms of these transfer functions as $\tilde{P}$ varies over Π. It is clearly desirable that we design a controller C that achieves such a uniform norm bound on all transfer functions. With respect to Figure 1, denote the map $(r,d) \mapsto (x_1,x_2)$ by T_{zw}, which clearly depends on $\tilde{P}$. Let the performance property of interest be the same as in Section 1. The following definition is in [2].

Definition 6.1. The controller C achieves *uniform stability for* Π if

$$\sup_{\tilde{P}\in\Pi} \left\| T_{zw}(\tilde{P}) \right\|_\infty < \infty.$$

We define uniform performance similarly.

Definition 6.2. The controller achieves *uniform performance for* Π if it achieves uniform stability and robust performance for Π.

When the supremum in the definitions is finite, we call it the *uniform bound.* It is clear from the definitions that uniform stability (performance) is a stronger notion than robust stability (performance). We now show by an example that these are indeed strictly stronger.

Example. Consider Example 3. We have shown that C achieves robust performance (and hence robust stability) for Π_M. Consider $\frac{C}{1+\tilde{P}C}$. For any $\omega \in \mathbb{R}^+$, we have

$$\left|\frac{C}{1+\tilde{P}C}\right| = \left|\frac{S}{1+\Delta W_2 T}\right| = \frac{\frac{\omega}{\sqrt{1+\omega^2}}}{|1+\Delta(i\omega)T(i\omega)|}.$$

We also have a sequence of ω_n for which $\frac{\omega_n}{\sqrt{1+\omega_n^2}}/(1-\frac{1}{\sqrt{1+\omega_n^2}})$ becomes arbitrarily large. For each ω_n, we can select a $\Delta_n \in RH_\infty \cap \Delta$ such that $|1+\Delta_n T(i\omega_n)| = 1-|T(i\omega_n)|$. For this sequence of Δ_n, we have

$$\lim_n \left\| \frac{S}{1+\Delta_n W_2 T}\right\|_\infty = \infty. \qquad \square$$

However, uniform stability need not always be stronger than robust stability. For the additive perturbation class $\Pi_A := \{\tilde{P} = P + \Delta W_2 : \|\Delta\|_\infty \le 1\}$, uniform stability (performance) is equivalent to robust stability (performance).

The uniform stability (performance) problem amounts to determining if a given controller C achieves uniform stability (performance) for a given uncertainty set. The following is easy to prove:

$$\|M\|_\infty < 1 \iff \sup_{\Delta\in\boldsymbol{\Delta}} \left\|(I+M\Delta)^{-1}\right\|_\infty < \infty.$$

Paraphrasing the right hand side of the above equivalence as 'uniform M-Δ stability', we may expect that uniform M-Δ instability does not imply uniform instability for multiplicative perturbations, if the perturbation set is the closed unit ball $\boldsymbol{\Delta}$. The following proposition shows that these notions are different even when the perturbation set is $\boldsymbol{\Delta}^o$.

Proposition 6.3. *Let C stabilize the nominal plant P. Then:*

(i) $\|W_2T\|_\infty < 1$ is sufficient but not necessary for C to achieve uniform stability for Π_M (Π_M^o).

(ii) $\| \, |W_1S| + |W_2T| \, \|_\infty < 1$ is sufficient but not necessary for C to achieve uniform performance for Π_M (Π_M^o).

Proofs of sufficiency are easy. We now show C can achieve uniform performance for Π_M even when $\|W_2T\|_\infty = 1$.

Example 4. Let $P = \frac{2s+1}{s^2}$, $C = 1$, $W_1 = 0.99$, $W_2 = \frac{1}{2s+1}$. Arguing as in Example 1, we can show that system is robustly stable for Π_M. For each ω and each $\Delta \in \boldsymbol{\Delta}$, we have

$$\left|\frac{W_1S}{1+\Delta W_2T}\right| \le \frac{|W_1S|}{|1-|W_2T||} = \frac{0.99\frac{\omega^2}{1+\omega^2}}{1-\frac{1}{1+\omega^2}} = 0.99.$$

So, C achieves robust performance for Π_M. Since C and W_1 are stable and invertible in H_∞, all four transfer functions are also uniformly

bounded. An upper bound is 2. Same conclusion, with the same bounds, is reached even with $W_2 = 1$.

□

With $\Delta \in \Delta^o$, we have seen in Proposition 5.1 that robust performance for Π_M^o is equivalent to robust stability for $\tilde{\Pi}^o$. We now examine if there is such an equivalence between uniform performance and uniform stability.

Proposition 6.4. *Let C stabilize P. Then:*
(i) C uniformly stabilizes $\tilde{\Pi}^o$ implies C achieves uniform performance for Π_M^o.
(ii) Converse of (i) is false.

Proof of (i) is easy. The following trivial example proves (ii).

Example 5. Let $P = C = 1, W_1 = 1/2, W_2 = 3/2$. Then $S = T = 1/2$. It is easily verified that C achieves uniform performance for Π_M^o. Choose $\Delta_1 = \Delta_2 = \epsilon - 1$. As ϵ goes to zero, $S\frac{1+\Delta_1 W_1}{1+\Delta_1 W_1 S+\Delta_2 W_2 T}$ becomes arbitrarily large. □

We have seen an uncertainty set for which uniform stability (performance) is strictly stronger than robust stability (performance), but we do not have a necessary and sufficient condition for uniform stability. We now consider another uncertainty set where uniform stability is strictly stronger than robust stability, but for which there is a necessary and sufficient condition for uniform stability. Consider

$$\Pi_A^o := \{\tilde{P} = P + \Delta W_2 : \Delta \in \Delta^o\},$$
$$\tilde{\Pi}_A^o := \{\frac{P + \Delta_2 W_2}{1 + \Delta_1 W_1} : \Delta_1, \Delta_2 \in \Delta^o, \Delta_1 \not\equiv -W_1^{-1}\}$$

where $W_1, W_2 \in RH_\infty$ are fixed. The following proposition is easy to prove.

Proposition 6.5. *Let C stabilize P. Then:*
(i) C robustly stabilizes Π_A^o if and only if $\|W_2 CS\|_\infty \leq 1$.
(ii) C uniformly stabilizes Π_A^o if and only if $\|W_2 CS\|_\infty < 1$.
(iii) $\| |W_1 S| + |W_2 CS| \|_\infty \leq 1$ if and only if C achieves robust performance for Π_A^o.
(iv) $\| |W_1 S| + |W_2 CS| \|_\infty < 1$ is sufficient but not necessary for C to achieve uniform performance for Π_A^o.
(v) C uniformly stabilizes $\tilde{\Pi}_A^o$ if and only if $\| |W_1 S| + |W_2 CS| \|_\infty < 1$.

We conclude from parts (iv) and (v) that C achieves uniform performance for Π_A^o does not imply C uniformly stabilizes $\tilde{\Pi}_A^o$.

7. Conclusions

With the perturbation class defined by the open unit ball $\mathbf{\Delta^0}$, we have a more elegant theory for robust stability and performance than when the perturbation class is defined by the closed unit ball $\mathbf{\Delta}$. The $M - \Delta$ analysis treats a slightly different problem than the one stated here. The difference is not merely due to closed right half-plane pole-zero cancellation in the perturbed plant, as is shown in the case of uniform stability. Even when the perturbation class is defined by the open unit ball, the condition for uniform stability (performance) is asymmetrical.

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Minimization of the L^∞-Induced Norm for Sampled-Data Systems*

Bassam Bamieh[†], Munther A. Dahleh[‡], J. Boyd Pearson[§]

Abstract

In this paper, a complete solution for the ℓ^1 sampled-data problem is furnished for arbitrary plants. The ℓ^1 sampled-data problem is described as follows: Given a continuous-time plant, with continuous-time performance objectives, design a digital controller that delivers this performance. This problem differs from the standard discrete-time methods in that it takes into consideration the inter-sampling behavior of the closed loop system. The resulting closed loop system dynamics consist of both continuous-time and discrete-time dynamics and thus such systems are known as *hybrid* systems. It is shown that given any degree of accuracy, there exists a standard discrete-time ℓ^1 problem, which can be determined apriori, such that for any controller that achieves a level of performance for the discrete-time problem, the same controller achieves the same performance within the prescribed level of accuracy if implemented as a sampled-data controller. This is accomplished by first converting the the hybrid system into an *equivalent* infinite dimensional discrete-time system using the lifting technique in continuous time, then the infinite dimensional parts of the system which model the inter-sample dynamics are approximated. This approximation is done independently of the controller, and explicit bounds are obtained for the degree of approximation. It is shown that the convergence of this approximation is at least as $\frac{1}{n}$.

1 Introduction

This paper is concerned with designing digital controllers for continuous-time systems to optimaly achieve certain performance specifications in the presence of uncertainty. Contrary to discrete time designs, such controllers are designed taking into consideration the inter-sample behavior of the system. Such hybrid systems are generally known as sampled-data systems, and have recently received renewed interest by the control community.

The difficulty in considering the continuous time behavior of sampled-data systems, is that it is time varying, even when the plant and the controller are both continuous-time and discrete-time time-invariant respectively. We consider in this paper the *standard problem with sampled-data controllers* (or the sampled-data problem, for short) shown in figure 1. The continuous time controller is constrained to be sampled-data controller, that is, it is of the form $\mathcal{H}_\tau C \mathcal{S}_\tau$. The generalized plant is continuous-time time-invariant and C is discrete-time time-invariant,$\mathcal{H}_\tau$ is a zero order hold (with period τ), and $\mathcal{S}_\tau$ is an ideal sampler (with period τ). $\mathcal{H}_\tau$ and $\mathcal{S}_\tau$ are assumed synchronized. Let $\mathcal{F}(G, \mathcal{H}_\tau C \mathcal{S}_\tau)$ denote the mapping between the exogenous input and the regulated output. $\mathcal{F}(G, \mathcal{H}_\tau C \mathcal{S}_\tau)$ is in general time varying, in fact it is τ-periodic where τ is the period of the sample and hold devices.

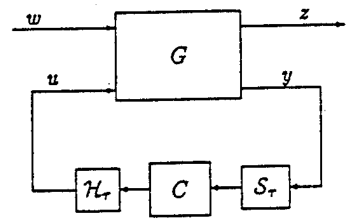

Figure 1: Hybrid discrete/continuous time system

Sampled-data systems have been studied by many researchers in the past in the context of LQG controllers (e.g. [19]). Recently, Chen and Francis [4] studied this problem in the context of $\mathcal{H}^\infty$ control, and were able to provide a solution in the case where the regulated output is in discrete time and the exogenous input is in continuous time. The exact problem was solved in [2],[3], and independently in [12] and [20]. The L^∞-induced norm problem (the one we are concerned with in this paper) was considered in [9].

In this paper we will use the framework developed in [2],[3], to study the ℓ^1 sampled-data problem. Precisely, the controller is designed to minimize the induced norm of the periodic system over the space of bounded inputs (i.e. L^∞). This minimization results from posing time domain specifications and design constraints, which is quite natural for control system design. To emphasize the point made earlier, the inputs are continuous time inputs, the errors are continuous time errors (see figure 1), however the system is a hybrid system with a continuous-time plant and a discrete-time controller. The discrete time method for ℓ^1 designs (e.g. [5],[15]), cannot handle this problem directly, and is only concerned with the performance at the sampling instants. The solution provided in this paper is to solve the sampled-data problem by solving an (almost) equivalent discrete time ℓ^1 problem. While this was the approach followed in [9], the main contribution of this paper is that it provides bounds that can be computed apriori to determine the equivalent discrete-time problem, given any desired degree of accuracy and thus provides a solution for the synthesis problem. The solution in this paper is presented in the context of the lifting framework of [2], [3], as an approximation procedure for certain infinite dimensional problems. This approach has the advantage of being more transparent than that in [9].

As already mentioned, sampled-data systems are periodic, the main theoretical tool we use for dealing with periodic systems is a *lifting* technique for continuous time systems devel-

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[†]Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801

[‡]Laboratory of Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA

[§]Dept. of Electrical and Computer Engineering, Rice University, Houston, TX 77030

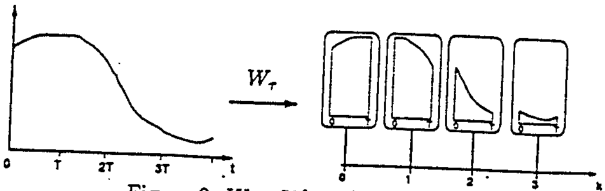

Figure 2: $W_\tau : L_e^p[0,\infty) \longrightarrow \ell_{L^p[0,\tau]}$

oped in [2], [3][1]. The technique establishes a strong correspondence between linear periodic systems and time invariant infinite dimensional systems. In the next section we briefly describe the lifting and it's application to the sampled-data problem. We then set up an equivalent infinite dimensional problem whose solution is obtained using an approximation procedure. Formulas for the (almost) equivalent discrete time problem are given in section 3. In the later sections the issue of the convergence of the approximation procedure is investigated, where the main result is a design inequality (5) which expresses the degree of approximation of the hybrid problem by a discrete-time problem, in terms of the dynamics of the plant and independently of the choice of the controller. This inequality is arrived at by decomposing the equivalent infinite dimensional problem and analyzing the decomposition. Space limitations preclude including the details of this derivation which are presented elsewhere [1].

2 The Lifting Technique

In this section we briefly summarize the lifting technique for continuous-time periodic systems developed in [2], [3], and apply it to the sampled-data problem. The idea of the lifting technique is to put a periodic continuous-time system in a strong correspondence with a shift-invariant (i.e. discrete-time time-invariant) system, which amounts to rearranging the original system so that its periodicity can be viewed as shift invariance. To accomplish this, we first define the lifting for signals, for which the appropriate signal spaces need to be established.

For continuous time signals, we consider the usual $L^\infty[0,\infty)$ space of essentially bounded functions [8], and it's extended version $L_e^\infty[0,\infty)$. We will also need to consider discrete time signals that take values in a function space, for this, we define ℓ_X to be the space of all X-valued sequences, where X is some Banach space. We define ℓ_X^∞ as the subspace of ℓ_X with bounded norm sequences, i.e. where for $\{f_i\} \in \ell_X$, the norm $\|\{f_i\}\|_{\ell_X^\infty} := \sup_i \|f_i\|_X < \infty$. Given any $f \in L_e^\infty[0,\infty)$, we define it's *lifting* $\hat{f} \in \ell_{L^\infty[0,\tau]}$, as follows: $\hat{f}$ is an $L^\infty[0,\tau]$-valued sequence, we denote it by $\{\hat{f}_i\}$, and for each i,

$$\hat{f}_i(t) := f(t + \tau i) \quad 0 \le t \le \tau.$$

The lifting can be visualized as taking a continuous time signal and breaking it up into a sequence of 'pieces' each corresponding to the function over an interval of length τ (see figure 2). Let us denote this lifting by $W_\tau : L_e^\infty[0,\infty) \longrightarrow \ell_{L^\infty[0,\tau]}$. W_τ is a linear isomorphism, furthermore, if restricted to $L^\infty[0,\infty)$, then $W_\tau : L^\infty[0,\infty) \longrightarrow \ell^\infty_{L^\infty[0,\tau]}$ is an isometry, i.e. it preserves norms.

Using the lifting of signals, one can define a lifting on systems. Let G be a linear continuous time system on $L_e^\infty[0,\infty)$, then it's *lifting* $\hat{G}$ is the discrete time system $\hat{G} := W_\tau G W_\tau^{-1}$, this is illustrated in the commutative diagram below:

[1] Essentially the same technique was arrived at independently in [20] and [21]

$$\begin{tikzcd} \ell_{L^\infty[0,\tau]} \arrow[r, "\hat{G}"] \arrow[d, "W_\tau^{-1}"'] & \ell_{L^\infty[0,\tau]} \\ L_e^\infty[0,\infty) \arrow[r, "G"] & L_e^\infty[0,\infty) \arrow[u, "W_\tau"'] \end{tikzcd}$$

Thus $\hat{G}$ is a system that operates on Banach space ($L^\infty[0,\tau]$) valued signals, we will call such systems infinite dimensional. Note that since W_τ is an isometry, if G is stable, i.e. a bounded linear map on L^∞ then $\hat{G}$ is also stable, and furthermore, their respective induced norms are equal, $\|\hat{G}\| = \|G\|$. The correspondence between a system and it's lifting also preserves algebraic system properties such as addition, cascade decomposition and feedback (see [2] for the details).

The usefulness of the lifting in the sampled-data problem is the fact that if G is a τ-periodic system, then $\hat{G}$ is commutes with the shift on $\ell_{L^\infty[0,\tau]}$, that is, $\hat{G}$ is shift-invariant. This basic fact allows us to treat continuous time periodic systems as discrete-time time-invariant, albeit infinite dimensional systems.

State space models can be found for the lifted systems. To illustrate, let G be a continuous-time time-invariant system given by a state space realization $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. In [2] it was shown that the lifting $\hat{G}$ has a state space realization given by:

$$\hat{G} \;=\; \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} e^{A\tau} & e^{A(\tau-\hat{s})}B \\ \hline Ce^{A\hat{t}} & Ce^{A(\hat{t}-\hat{s})}1(\hat{t}-\hat{s})B + D\delta(\hat{t}-\hat{s}) \end{array}\right] \quad (1)$$

$$\begin{aligned} \hat{B} &: L^\infty[0,\tau] \longrightarrow \mathbb{R}^x \\ \hat{A} &: \mathbb{R}^x \longrightarrow \mathbb{R}^x \\ \hat{C} &: \mathbb{R}^x \longrightarrow L^\infty[0,\tau] \\ \hat{D} &: L^\infty[0,\tau] \longrightarrow L^\infty[0,\tau] \end{aligned}$$

where the operators $\hat{C}, \hat{B}, \hat{D}$ are given in terms of their kernel functions, and $1(.)$ is the unit step function.

Notation: It simplifies the notation greatly to use the same symbol for an operator and its kernel, for example, $\hat{D}(t,s)$ (or $\hat{B}(s)$) refer to the kernel functions representing the operator $\hat{D}$ (or $\hat{B}$). For operators that map a function space to $\mathbb{R}^n$, such as $\hat{B}$ above, we generally use s (or $\hat{s}$) to denote the variable of the kernel function, and for operators that map $\mathbb{R}^n$ to a function space such as $\hat{C}$ above, we use the variable t (or $\hat{t}$). The kernel representation for the operators $\hat{B}, \hat{C}, \hat{D}$ means that their action is given by

$$\hat{B}\,u \;=\; \int_0^\tau \hat{B}(\hat{s})\, u(\hat{s})\, d\hat{s} \qquad \hat{C}\,x \;=\; \hat{C}(\hat{t})x\,, \quad \hat{t} \in [0,\tau]$$

$$(\hat{D}u)(\hat{t}) \;=\; \int_0^\tau \hat{D}(\hat{t},\hat{s})\, u(\hat{s})\, d\hat{s}$$

Note that the state space of $\hat{G}$ is finite dimensional (the x in $\mathbb{R}^x$ refers to the dimension of the state space of G), while it's input and output spaces are infinite dimensional. This fact is significant in that, although lifted systems have infinite dimensional input and output spaces, they can be realized with a state space of dimension no larger than the dimension of the original continuous-time state space model.

To apply the lifting to the sampled-data problem, consider again the standard problem of figure 1, and denote the closed loop operator by $\mathcal{F}(G, \mathcal{H}_\tau C \mathcal{S}_\tau)$. Since the lifting is an isometry, we have that $\|\mathcal{F}(G, \mathcal{H}_\tau C \mathcal{S}_\tau)\| = \|W_\tau \mathcal{F}(G, \mathcal{H}_\tau C \mathcal{S}_\tau) W_\tau^{-1}\|$, this is shown in figure 3(a). In figure 3(b), we lump the lifting operators W_τ and W_τ^{-1} and the sample and hold operators and consider a new generalized plant $\tilde{G}$. $\tilde{G}$ is a discrete time system with one infinite dimensional input and output (corresponding

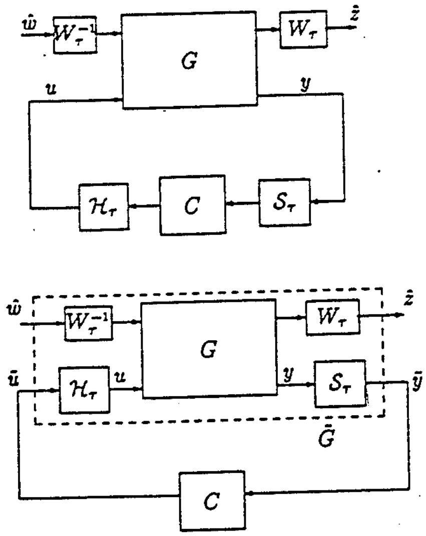

Figure 3: Equivalent Problem

to $\hat{w}$ and $\hat{z}$) and one finite dimensional input and output (corresponding to $\tilde{u}$ and $\tilde{y}$). Thus, $\mathcal{F}(\tilde{G},C) = W_\tau \mathcal{F}(G,\mathcal{H}_\tau C\mathcal{S}_\tau)W_\tau^{-1}$, which means that the closed loop operator $\mathcal{F}(\tilde{G},C)$ is in fact the lifting of the closed loop operator $\mathcal{F}(G,\mathcal{H}_\tau C\mathcal{S}_\tau)$. Since the lifting W_τ is an isometry, we have then characterized the L^∞ induced norm of the hybrid system as the $\ell^\infty_{L^\infty[0,\tau]}$ induced norm of the time invariant system $\mathcal{F}(\tilde{G},C)$. The conclusion is that the problem of minimizing the L^∞ induced norm of the sampled-data system, is equivalent to that of minimizing the induced norm of the infinite dimensional but time-invariant system $\mathcal{F}(\tilde{G},C)$. The previous discussion together with the characterization of internal stability for hybrid systems in [11] (conditions for non-pathological sampling) yields the following theorem:

Theorem 1 *Let G and $\tilde{G}$ be as in figure 3, then for any finite dimensional C*

(i) $\mathcal{F}(G,\mathcal{H}_\tau C\mathcal{S}_\tau)$ is internally stable if and only if $\mathcal{F}(\tilde{G},C)$ is.

(ii) $\|\mathcal{F}(G,\mathcal{H}_\tau C\mathcal{S}_\tau)\| = \|\mathcal{F}(\tilde{G},C)\|$.

This reformulation of the sampled-data problem to the problem with $\tilde{G}$ has several advantages, first, the controller has no 'structural constraints' on it, in contrast to the previous formulation where the controller is constrained to be a sampled-data controller, i.e. of the form $\mathcal{H}_\tau C\mathcal{S}_\tau$, second, both the controller C and the generalized plant $\tilde{G}$ are shift-invariant, thus the periodicity of the original system is 'removed', and third, all parts of the system are operating over the same time set (discrete-time). The price paid for these advantages is the infinite dimensionality of the input and output spaces. In this paper we will show how one can reduce the problem to a finite dimensional one by 'approximating' the input and output spaces by finite dimensional spaces, thus reducing the problem to a standard finite dimensional ℓ^1 problem.

We now present (from [2]) a state space realization for the new generalized plant $\tilde{G}$ which will be useful in studying the problem further. Let the original continuous time plant G be given by the following realization

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right].$$

It is assumed that the sampler is preceded with a presampling filter which is a strictly causal linear system, this is a realistic assumption since an ideal sampler is not a practical device, a real sampler can be modeled as an integrator with a fast time constant followed by and ideal sampler. The system shown above represents a generalized plant with the presampling filter absorbed in it, the fact that $D_{21} = D_{22} = 0$ is due to the strict causality of the presampling filter, this also guarantees that the ideal sampler only operates on continuous signals. It can be shown ([2]) that a realization for the generalized plant $\tilde{G}$ (figure 3) is given by

$$\tilde{G} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \tilde{B}_2 \\ \hline C_1 & D_{11} & D_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right] =$$

$$\left[\begin{array}{c|cc} e^{A\tau} & e^{A(\tau-s)}B_1 & \Psi(\tau)B_2 \\ \hline C_1e^{At} & C_1e^{A(t-s)}1\!\!1_{(t-s)}B_1 + D_{11}\delta(t-s) & C_1\Psi(t)B_2 + D_{12} \\ C_2 & 0 & 0 \end{array}\right]$$

where $\Psi(t) := \int_0^t e^{As}ds$. The system $\tilde{G}$ has the following input and output spaces

$$\begin{aligned} \tilde{G}_{11}:&\ \ell_{L^\infty[0,\tau]} \longrightarrow \ell_{L^\infty[0,\tau]} \\ \tilde{G}_{12}:&\ \ell_{R^u} \longrightarrow \ell_{L^\infty[0,\tau]} \\ \tilde{G}_{21}:&\ \ell_{L^\infty[0,\tau]} \longrightarrow \ell_{R^y} \\ \tilde{G}_{22}:&\ \ell_{R^u} \longrightarrow \ell_{R^y} \end{aligned}$$

3 Solution Procedure

Using the lifting we are able to convert the problem of finding a controller to minimize the L^∞ induced norm of the hybrid system (figure 1) into the following standard problem with an infinite dimensional generalized plant $\tilde{G}$:

$$\mu := \inf_{C \text{ stablizing}} \|\mathcal{F}(G,\mathcal{H}_\tau C\mathcal{S}_\tau)\| = \inf_{C \text{ stablizing}} \|\mathcal{F}(\tilde{G},C)\| \tag{2}$$

We also note that because of theorem 1, suboptimal solutions to the above problem will also be suboptimal (with the same norm) for the hybrid system.

The above infinite dimensional problem is solved by an approximation procedure through solving a standard MIMO ℓ^1 problem. The idea we use is similar to those in [9] and [13] where multirate sampling is used to obtain discrete-time systems that approximate the continuous time behavior of hybrid systems. This approximation procedure was used in [9] to address the ℓ^1 sampled-data problem. The approximation procedure we use is essentially equivalent to that in [9], however, since we introduce it directly as an approximation to the lifted problem (2), the nature of the approximation is more transparant and we are able to explicitly isolate the parts of the system that need to be approximated independently of the controller. The consequence is that we are able to obtain explicit bounds on the degree of approximation in terms of constants that can be computed *apriori*, and that are dependent only on the plant.

We now describe the approximation procedure. Let $\mathcal{H}_n$ and $\mathcal{S}_n$ be the following operators defined between $L^\infty[0,\tau]$ and $\ell^\infty(n)$ ($\ell^\infty(n)$ is R^n with the maximum norm),

$$\mathcal{S}_n: \ L^\infty[0,\tau] \longrightarrow \ell^\infty(n) \quad (\mathcal{S}_n u)(i) = u(\frac{\tau}{n}i)\ ;\ u \in L^\infty[0,\tau]$$

$$\mathcal{H}_n: \ \ell^\infty(n) \longrightarrow L^\infty[0,\tau] \quad (\mathcal{H}_n u)(t) = u(\lfloor\frac{tn}{\tau}\rfloor);\ \{u(i)\} \in \ell^\infty(n),$$

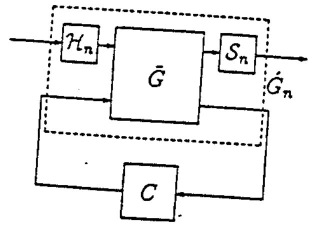

Figure 4: The system $\acute{G}_n$

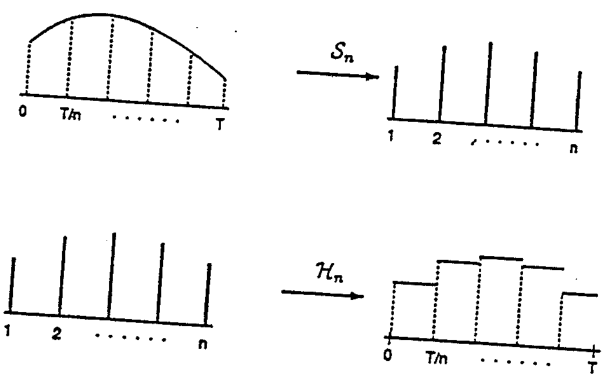

Figure 5: The operators $\mathcal{S}_n$ and $\mathcal{H}_n$

(strictly speaking, $\mathcal{S}_n$ is not an operator on $\mathcal{L}^\infty$ but on the subspace of left and right continuous functions, this distinction is irrelevant here since in our setting, assumptions are made to guarantee that $\mathcal{S}_n$ operates only on continuous signals), the above operators can be thought of as 'fast' sample and hold operators (see figure 5).

Now to approximate the infinite dimensional problem, we use the approximate closed loop system $\mathcal{S}_n\mathcal{F}(\bar{G},C)\mathcal{H}_n$ (see figure 4), and for each n we define

$$\mu_n := \inf_{C \text{ stabilizing}} \|\mathcal{S}_n\mathcal{F}(\bar{G},C)\mathcal{H}_n\|, \tag{3}$$

This new problem now involves the induced norm over $\ell^\infty_{\ell^\infty(n)}$, i.e. it is a standard MIMO ℓ^1 problem.

Let us denote the generalized plant associated with $\mathcal{S}_n\mathcal{F}(\bar{G},C)\mathcal{H}_n$ by $\acute{G}_n$, such that

$$\mathcal{S}_n\mathcal{F}(\bar{G},C)\mathcal{H}_n = \mathcal{F}(\acute{G}_n,C),$$

where $\acute{G}_n$ and a realization for it is given by,

$$\acute{G}_n := \begin{bmatrix} \mathcal{S}_n & 0 \\ 0 & I \end{bmatrix} \bar{G} \begin{bmatrix} \mathcal{H}_n & 0 \\ 0 & I \end{bmatrix} =$$
$$\left[\begin{array}{c|cc} \hat{A} & \hat{B}_1\mathcal{H}_n & \bar{B}_2 \\ \hline \mathcal{S}_n\bar{C}_1 & \mathcal{S}_n D_{11}\mathcal{H}_n & \mathcal{S}_n D_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right] =: \left[\begin{array}{c|cc} \hat{A} & \acute{B}_1 & \bar{B}_2 \\ \hline \acute{C}_1 & \acute{D}_{11} & \acute{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right].$$

The new operators, which are now matrices, are computed to be

$$\acute{C}_1 = \begin{bmatrix} C_1 \\ C_1 e^{A\tau/n} \\ \vdots \\ C_1\left(e^{A\tau/n}\right)^{n-1} \end{bmatrix}, \quad \acute{D}_{12} = \begin{bmatrix} D_{12} \\ C_1\Psi(\tau/n)B_2 + D_{12} \\ \vdots \\ C_1\Psi(\frac{\tau(n-1)}{n})B_2 + D_{12} \end{bmatrix},$$

$$\acute{B} = \Psi(\tau/n)\begin{bmatrix} B_1 & e^{A\tau/n}B_1 & \cdots & \left(e^{A\tau/n}\right)^{n-1}B_1 \end{bmatrix},$$

$$\acute{D}_{11} = \left\{\left[\begin{array}{c|c} e^{A\tau/n} & \Psi(\tau/n)B_1 \\ \hline C_1 & D_{11} \end{array}\right]\right\}_n,$$

where $\{.\}_n$ means the first $n \times n$ elements of the impulse response matrix of the discrete-time system given by the realization in $\{.\}$.

The solution to the original infinite dimensional problem (and thus to the sampled-data problem) is as follows: n can be chosen large enough such that if the designed controller C_n is almost optimal for the approximate problem (3), then it is almost optimal for the original problem (2). In essence, this approximation scheme 'converges', i.e. one can obtain almost optimal controllers by choosing n large enough and solving a MIMO ℓ^1 problem. Exactly what convergence means here is described next.

4 Design Bounds

In this section we investigate the nature of the approximation of $\|\mathcal{F}(\bar{G},C)\|$ by $\|\mathcal{F}(\acute{G}_n,C)\|$. In order to show that the synthesis procedure outlined in the previous section yields controllers with performance arbitrarily close to the optimal, one needs to obtain explicit bounds on the degree of approximation of $\|\mathcal{F}(\bar{G},C)\|$ by $\|\mathcal{F}(\acute{G}_n,C)\|$.

Let us begin with analysis. Note that since $\|\mathcal{F}(\bar{G},C)\|$ is a periodically time varying system, its $\mathcal{L}^\infty$-induced norm is not immediately computable. An alternative method of computing $\|\mathcal{F}(\bar{G},C)\|$ comes from the limit

$$\|\mathcal{F}(\bar{G},C)\| = \lim_{n\to\infty}\|\mathcal{S}_n\mathcal{F}(\bar{G},C)\mathcal{H}_n\| =: \lim_{n\to\infty}\|\mathcal{F}(\acute{G}_n,C)\|, \tag{4}$$

for a fixed C. This formula can be proved using arguments about the approximation of continuous functions by simple functions in $\mathcal{L}^\infty$ ([17]). Since $\mathcal{F}(\acute{G}_n,C)$ is a time-invariant MIMO system and $\|\mathcal{F}(\acute{G}_n,C)\|$ is its ℓ^1 norm, it can be computed to any desired accuracy, consequently, by (4) the actual norm, $\|\mathcal{F}(\bar{G},C)\|$ can be computed to any desired accuracy. However, equation (4) is by far not sufficient to show the convergence of the synthesis procedure, since given only (4), the rate of convergence may depend on the choice of C.

Our objective is to obtain explicit bounds on $\|\mathcal{F}(\bar{G},C)\|$ in the following form

Main Inequality : *There are constants K_o and K_1 which depend only on G, such that*

$$\|\mathcal{F}(\bar{G},C)\| \leq \frac{K_1}{n} + \left(1+\frac{K_o}{n}\right)\|\mathcal{F}(\acute{G}_n,C)\|, \tag{5}$$

The significance of the bound (5) is that it is exactly what is needed for synthesis. When one performs an ℓ^1 design on $\acute{G}_n$, the result is a controller that keeps $\|\mathcal{F}(\acute{G}_n,C)\|$ small, but the objective is to keep the $\mathcal{L}^\infty$-induced norm of the hybrid system (or equivalently $\|\mathcal{F}(\bar{G},C)\|$) small, and the inequality (5) guarantees this.

To be more precise, first note that we can immediately obtain

$$\|\mathcal{F}(\acute{G}_n,C)\| \leq \|\mathcal{F}(\bar{G},C)\| \quad \forall n,$$

since

$$\begin{aligned} \|\mathcal{F}(\acute{G}_n,C)\| &= \|\mathcal{S}_n\mathcal{F}(\bar{G},C)\mathcal{H}_n\| \leq \|\mathcal{S}_n\|\,\|\mathcal{F}(\bar{G},C)\|\,\|\mathcal{H}_n\| \\ &\leq \|\mathcal{F}(\bar{G},C)\|, \end{aligned}$$

because $\|\mathcal{H}_n\| \leq 1$ on $\ell^\infty(n)$ and $\|\mathcal{S}_n\| \leq 1$ on the subspace of $\mathcal{L}^\infty$ for which it is defined. The above inequality immediately implies that $\mu_n \leq \mu$. The synthesis procedure is guided by the following; for a fixed n, if one performs a MIMO ℓ^1 design (as in [15]) on $\acute{G}_n$ and obtains a $\mu_n+\epsilon$ optimal controller (given by

C_n), i.e. $\|\mathcal{F}(\acute{G}_n, C)\| \leq \mu_n + \epsilon$, then inequality (5) provides that if C_n is implemented in the hybrid system, then

$$\begin{aligned}\|\mathcal{F}(G, \mathcal{H}_\tau C_n S_\tau)\| = \|\mathcal{F}(\tilde{G}, C_n)\| &\leq \frac{K_1}{n} + \left(1 + \frac{K_o}{n}\right)\|\mathcal{F}(\acute{G}_n, C)\| \\ &\leq \frac{K_1}{n} + \left(1 + \frac{K_o}{n}\right)(\mu_n + \epsilon) \\ &\leq \frac{K_1}{n} + \epsilon\left(1 + \frac{K_o}{n}\right) + \mu\left(1 + \frac{K_o}{n}\right). \end{aligned} \tag{6}$$

Therefore, if a controller with a level of performance of $\mu + \delta$ is required (for any $\delta > 0$), we simply choose n and ϵ *apriori* such that the right hand side of (6) is bounded by $\mu + \delta$.

It is worthwhile noting that the problem of minimizing $\|\mathcal{F}(\acute{G}_n, C)\|$ is immediately a standard ℓ^1 problem with time-invariant plant. Hence, there is no need to apply any further lifting on the problem, which contrasts the approach in [9]. Also, we note that eventhough the approximation problem is in fact a multirate sampled problem, it reflects no structural constraints on the controller. General multirate sampled problem do not share this property (see [7]).

In the derivation of the main inequality 5, several interesting issues come up, and the bounds on the approximation is obtained by dissecting the infinite dimensional system $\tilde{G}$ closely. We refere the interested reader to [1] for the full discussion.

5 Conclusions

This paper provides a complete solution for the sampled-data ℓ^1 problem through approximation. Utilizing lifting techniques, the input/output map is decomposed in a such a way that the infinite-dimensional part of the system is isolated independently of the controller. This part is then approximated in a precise way by a finite dimensional system, whose dimension can be determined given any degree of accuracy. Computable bounds on the norm of the difference of the actual system and the approximated system are furnished, and they all depend entirely on the system's data.

It is interesting to note that the same approach can be followed to solve the problem for the L^1-induced norm, then, by a simple convexity argument, a solution for the general L^p-induced norm can be obtained. However, the case of L^2 induced norm admits a cleaner solution [2], and an exact discrete-time problem can be obtained.

The approach followed in this paper is readily applicable to the structured perturbations problem for sampled-data systems [14]. The minimization problem in this set-up involves spectral radius functions, and a similar result follows from the continuity of the spectral radius function. The derivation of explicit bounds takes more work and will be reported elsewhere.

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Behavior is more fundamental than representations*†

A. P. Kishore J. B. Pearson

Department of Electrical and Computer Engineering
Rice University, Houston, TX 77251-1892, U.S.A.

A input-output system is a relation between two function spaces. The classical input-output framework treats a system as a *map* between function spaces. The graph of this map, which is the collection of all compatible input-output pairs, constitutes the behavior of the system [3]. The behavior of a system can sometimes admit a behavioral equation representation such as a kernel representation or a difference equation representation. Such a representation, when it exists, may not be unique. Given a representation with a certain structure (for instance, a lower triangular kernel) it is usually easily shown that the represented behavior has a corresponding property (non-anticipation). However, if the behavior has a property (say, non-anticipation), representations of the behavior may not have the corresponding structure (lower triangularity). Therefore, representations are of secondary importance to behaviors. It is the behavior that is fundamental; not its representation [3].

We will illustrate these points with kernel representations, concentrating on systems operating on one-sided discrete-time signals in the sequence space ℓ. We say that $G : \mathcal{D}(G) \subseteq \ell \to \mathcal{R}(G) \subseteq \ell$ has *a kernel representation* if there exists a $g : Z_+ \times Z_+ \to \mathbf{R}$ such that for all nonnegative integers n

$$(Gu)(n) = \sum_{m=0}^{\infty} g(n,m)u(m), \quad \forall u \in \mathcal{D}(G).$$

Not all linear systems have a kernel representation [2]. We first point out that compactness of the map is neither sufficient [1] nor necessary for kernel representation.

Even when a system has a kernel representation, the representation may not be unique. This is shown by an example of a linear shift-invariant nonanticipatory system that has infinitely many kernel representations. Out of the infinitely many representations for this system, one is lower-triangular and one is upper-triangular. Therefore, non-anticipation is a property of a system and is not necessarily a (structural) prop-

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erty of its representation. Some kernel representations of this system have Toeplitz structure, and some do not. Shift-invariance is then a property of a system that may or may not be reflected in the structure of its representation.

We then point out that boundedness of a map may not be reflected in the structure of its kernel representation (or in a minimal state-space representation).

Since properties such as shift-invariance, non-anticipation, and boundedness are properties of a system and are not necessarily structural properties of a representation of the system (unless the representation is unique), a system is a logically distinct object from its representation. It is the behavior of the system that needs to be examined for properties of interest, and not the structure of a representation of the system.

We argue the above points using traditional definitions of linearity, shift-invariance, nonanticipation, and boundedness. We will make a case for nonstandard definitions of linearity and shift-invariance. The main practical reason for studying linear mathematics is that local behavior of a nonlinear map is often linear. That is, if we restrict the domain of a given non-linear map, the restricted map (the *restriction*) may become linear, thereby making analysis easier. Then, if we were to restrict the domain further, we would like the resulting restriction to be still linear. Considering that linearity is an analytically desirable property of a map, we would like all the restrictions of a linear map to inherit this property. Similarly, inheritance by restrictions is desirable with respect to shift-invariance, non-anticipation, and continuity, from a practical point of view.

However, in the classical framework for input-output systems, linearity and shift-invariance are not inherited by restrictions, while continuity is. For example, the linearity or shift-invariance of the identity map depends on whether or not its input class is linear or shift-invariant. However, it is the behavior of a system in a given configuration that is more important than the properties of the domain of its definition. As far as possible, it is the behavior of a system that we should focus our attention on; not the properties of its domain. Taking cue from a definition of non-anticipation in the classical framework [4], we propose new definitions for linearity and shift-invariance.

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Formulation of l_1 Optimal Control Problems without Interpolation *

J. S. McDonald and J. B. Pearson
Rice University
Houston, TX

Abstract

The solution of many l_1 problems requires Smith decompositions of l_1 matrices. In this note, we describe a class of problems, including many practical problems, for which this is not true. We also show generally how to pose and obtain approximate solutions to l_1 problems without Smith decomposition.

1 Introduction

The l_1 problem was formulated in [1] and investigated in, e.g., [2, 3, 4], in which it is posed as a linear program (LP) whose constraint matrix is constructed, in part, from Smith decompositions of two l_1 matrices. This decomposition does not exist for all l_1 matrices, and its computation is numerically unstable in any case. Thus it is of interest to find problems and/or solution methods which do not require it.

In this note, we describe a class of such problems which includes many practical problems. We also describe a class of semi-norm minimization problems which can be solved without Smith decomposition, and into which *every* l_1 problem can be embedded. We then use the embedding to obtain, under certain conditions, an infimizing sequence of sub-optimal solutions by solving finite LPs. In Section 2 we briefly state the l_1 problem, the main results are in Section 3, and Section 4 offers some conclusions.

$l_1^{m\times n}$ denotes $m \times n$ matrices with elements in the commutative domain l_1 and $\mathcal{F}l_1^{m\times n}$ denotes matrices with elements in its fraction field. Superscripts are dropped when there is no loss of clarity. A script letter denotes a causal discrete-time convolution system, and a capital (Roman) letter its impulse response matrix. Matrix multiplication is defined via convolution.

2 The l_1 Problem

Figure 1 shows the setting of the l_1 problem. $\mathcal{G}$ has inputs w (an n_w-vector of exogeneous inputs) and u (an n_u-vector of controls) and outputs z (an n_z-vector of errors) and y (an n_y-vector of measurements). G can be partitioned in the obvious way:

$$G = \begin{bmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix}$$

For stabilizable $\mathcal{G}$ with $G_{yu} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ (coprime over l_1), the l_1 problem is:

$$OPT(\mathcal{G}): \qquad \inf_{K\in\mathcal{S}_{U,V}} \|H - K\|_1 \ =: \ \mu_{OPT(\mathcal{G})}$$

where $\mathcal{S}_{U,V} := \{K \in l_1 : K = UQV,\ Q \in l_1\}$, $H := G_{zw} + G_{zu}M\tilde{Y}G_{yw}$, $U := G_{zu}M$, and $V := \tilde{M}G_{yw}$ are in l_1, and the following Bezout identity is satisfied:

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Note: H, U, and V depend on $\mathcal{G}$. We will show this dependence via bars, tildes, etc. (e.g., $\bar{\mathcal{G}} \leftrightarrow \bar{H}, \bar{U}, \bar{V}$).

3 Main Results

Theorem 1 *If $U, V \in l_1$ have left and right inverses, respectively, in l_1 and $U = N_U M_U^{-1}$, $V = \tilde{M}_V^{-1}\tilde{N}_V$ (coprime over l_1), then $\exists$ l_1 matrices satisfying*

$$\begin{bmatrix} N_U^{-L} \\ N_U^{\perp} \end{bmatrix} \begin{bmatrix} N_U & N_U^c \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} \tilde{N}_V \\ \tilde{N}_V^c \end{bmatrix} \begin{bmatrix} \tilde{N}_V^{-R} & \tilde{N}_V^{\perp} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and $K \in \mathcal{S}_{U,V}$ if and only if $K \in l_1$ and

$$\begin{bmatrix} N_U^{-L} \\ N_U^{\perp} \end{bmatrix} K \begin{bmatrix} \tilde{N}_V^{-R} & \tilde{N}_V^{\perp} \end{bmatrix} = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \qquad (1)$$

where $*$ denotes an irrelevant block.

Condition (1) does not refer to Smith decomposition and hence none is required for LP formulation.

*This research was supported by grants NSF ECS-8914467 and AFOSR-91-0036.

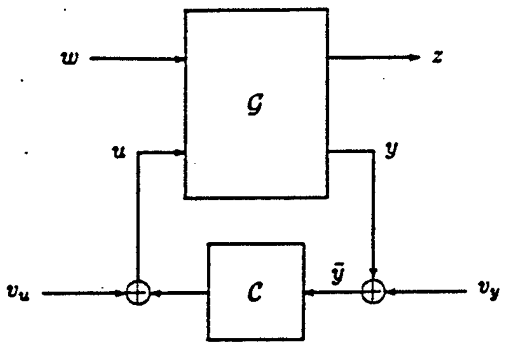

Figure 1: Standard Problem Setting

In many practical problems, v_u and v_y of Figure 1 correspond to actual disturbances and l_∞ norm constraints are required on u and $\tilde{y}$ to avoid saturation. In such problems, w, z, U, and V can be partitioned

$$w = \begin{bmatrix} \bar{w} \\ v_u \\ v_y \end{bmatrix}, \qquad z = \begin{bmatrix} \bar{z} \\ u \\ \tilde{y} \end{bmatrix}$$

$$U = \begin{bmatrix} MG_{zu} \\ M \\ N \end{bmatrix}, \quad V = \begin{bmatrix} G_{yw}\tilde{M} & \tilde{N} & \tilde{M} \end{bmatrix}$$

Thus U and V satisfy the hypotheses of Theorem 1. (This remains true if unimodular l_1 weights are introduced on $v_u, v_y, \tilde{y}, u$.)

We now define, given $\mathcal{G}$ and integers $n_o \leq n_z, n_i \leq n_w$, a semi-norm minimization problem

$OPTS(\mathcal{G}, n_o, n_i)$:

$$\inf_{K \in \mathcal{S}_{U,V}} \|\mathcal{P}_{n_o n_i}(H - K)\|_1 =: \mu_{OPTS(\mathcal{G}, n_o, n_i)}$$

where $\mathcal{P}_{n_o n_i} : l_1^{n_z \times n_w} \mapsto l_1^{n_o \times n_i}$ is a linear projection defined by $(\mathcal{P}_{n_o n_i} T)_{mn}(k) := T_{mn}(k)\ \forall m, n, k$.

Theorem 2 *If $\mathcal{G}$ is stabilizable, then there exist integers $n_U \leq n_u$, $n_V \leq n_y$ and a stabilizable $\bar{\mathcal{G}}$ with $\bar{G} \in \mathcal{F}l_1^{(n_z+n_U+n_y)\times(n_w+n_V+n_u)}$ such that $\bar{U}$ and $\bar{V}$ satisfy the hypotheses of Theorem 1 and*

1. *Given $K \in \mathcal{S}_{U,V}$, define $\bar{K} := \bar{U}Q\bar{V}$ for any $Q \in l_1$ such that $K = UQV$. Then $\bar{K} \in \mathcal{S}_{\bar{U},\bar{V}}$ and $\|\mathcal{P}_{n_z n_w}(\bar{H} - \bar{K})\|_1 = \|H - K\|_1$.*
2. *Given $\bar{K} \in \mathcal{S}_{\bar{U},\bar{V}}$, $K := \mathcal{P}_{n_z n_w}\bar{K} \in \mathcal{S}_{U,V}$ and $\|H - K\|_1 = \|\mathcal{P}_{n_z n_w}(\bar{H} - \bar{K})\|_1$.*

Thus *every* problem $OPT(\mathcal{G})$ can be embedded in a problem $OPTS(\bar{\mathcal{G}}, n_z, n_w)$ for a (larger) $\bar{\mathcal{G}}$ such that $\bar{U}$ and $\bar{V}$ satisfy the hypotheses of Theorem 1, and feasible solutions of $OPT(\mathcal{G})$ correspond to feasible solutions of $OPTS(\bar{\mathcal{G}}, n_z, n_w)$ of the same cost.

$OPTS(\bar{\mathcal{G}}, n_z, n_w)$ is generally infinite dimensional so we next define, given $\mathcal{G}$ and integers $n_o \leq n_z$, $n_i \leq n_w$, and n, an optimization problem

$OPTS(\mathcal{G}, n_o, n_i, n)$:

$$\inf_{\mathcal{S}_{U,V,n}} \|\mathcal{P}_{n_o n_i}(H - K)\|_1 =: \mu_{OPT(\mathcal{G}, n)}$$

$$\mathcal{S}_{U,V,n} := \{K \in \mathcal{S}_{U,V} : \mathrm{supp}(H - K) \subset \{0, \ldots, n\}\}$$

Theorem 3 *If $\mathcal{G}$ is stabilizable, $\exists K_N \in \mathcal{S}_{U,V}$ such that $supp(H - K_N) \subset \{0, \ldots, N\}$, and the finitely supported matrices are dense in $\mathcal{S}_{U,V}$, then $OPTS(\mathcal{G}, n_o, n_i, n)$ has optimal solutions $\forall n \geq N$, and $\mu_{OPTS(\mathcal{G}, n_o, n_i, n)} \searrow \mu_{OPTS(\mathcal{G}, n_o, n_i)}$ as $n \to \infty$.*

Every feasible solution of $OPTS(\mathcal{G}, n_o, n_i, n)$ is feasible for $OPTS(\mathcal{G}, n_o, n_i)$, so a sequence of optimal solutions for increasing n forms an infimizing sequence of feasible solutions for $OPTS(\mathcal{G}, n_o, n_i)$. For each n, $OPTS(\mathcal{G}, n_o, n_i, n)$ is a finite LP [5].

4 Conclusions

Theorem 1 shows that l_1 problems with invertible U and V can be formulated as LPs without Smith decomposition. Such problems arise, e.g., when both sensor and actuator are noisy and subject to saturation. Theorem 2 shows that *all* l_1 problems can be embedded in larger semi-norm minimization problems without need of Smith decomposition. Theorem 3 shows that, under certain conditions, a sequence of finite LPs can be solved to obtain performance arbitrarily close to optimal. The price of embedding is increased dimensionality; thus an obvious question for further research is how to embed in a problem of least dimension.

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Control System Design to meet Weighted l_∞ Specifications *

J. S. McDonald and J. B. Pearson

Department of Electrical and Computer Engineering

Rice University, Houston, Texas

e-mail: jbp@argo.rice.edu, Tel. (713) 527-8101 x3566

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Abstract

Two performance specifications based on l_∞ measures of weighted disturbance and error signals are defined. Both allow the treatment of magnitude, rate, and acceleration bounds on disturbances and errors. One is an incremental weighted specification which requires error signals to satisfy a constraint for as long (in time) as the disturbance satisfies a similar constraint. The other is a weighted specification which considers only disturbances satisfying a constraint for all time, and requires that errors do as well.

Notions of stability and system gain are defined corresponding to each specification and the gains are shown to be different. For the incremental specification, gain can be computed by solving a standard l_1 synthesis problem, and for the weighted specification a modified version of l_1 synthesis can be used.

It is shown how to formulate the synthesis problem corresponding to each specification as a linear program similar to the one arising in l_1 synthesis.

Keywords: l_1 optimal control, weighting functions, performance analysis, performance synthesis

Classification: Mini-Symposium M-4, "Robust Control Design"; 20 minute oral presentation

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Notation and Assumptions

Z and Z_+ denote the integers and the non-negative integers, respectively. D is the open unit disk in the complex plane. l_∞ and l_1 denote the classical sequence spaces defined on Z_+, and $l_1(Z)$ is the counterpart of l_1 defined on all of Z, i.e., the set of absolutely summable two-sided sequences. l_1 will be regarded as being embedded in $l_1(Z)$, i.e., as the subspace of $l_1(Z)$ supported on Z_+. Matrices will be referred as belonging to l_1 or $l_1(Z)$ and signals as belonging to l_∞, meaning their elements belong to those spaces. For notational convenience we define a space A of all z-transforms of sequences in $l_1(Z)$ with norm defined $\|\hat{H}\|_A := \|H\|_1$.

Throughout the paper, signals are vector sequences denoted by lower case letters (e.g., x). Systems are causal MIMO discrete time systems with convolution representations and are denoted by calligraphic letters (e.g., $\mathcal{H}$). Their impulse response matrices are denoted by corresponding upper case Roman letters (e.g., H), and their transfer function matrices by hatted letters (e.g., $\hat{H}$) where the z-transform is defined with z as the delay. A product (GH) of impulse response matrices means convolution $(G * H)$.

1 Introduction

Our standard problem setting is depicted in Figure 1. The generalized plant $\mathcal{G}$ has two inputs and two outputs. w is the disturbance input and is present for the purpose of modelling exogeneous inputs to the system (e.g., disturbances, measurement errors, etc.). u is the control input. z is the regulated output and consists of error signals which are to be minimized, and y is the measured output. The compensator $\mathcal{C}$ determines the control input u given the measured output y. $\mathcal{C}$ is to be chosen to internally stabilize the system and satisfy, if possible, other specifications.

The simplest l_∞ design problem is disturbance rejection, in which the specification is

Disturbance Rejection Specification:

- $w \in l_\infty$ and $\|w\|_\infty \leq 1$ implies $z \in l_\infty$ and $\|z\|_\infty \leq 1$.

$\|G\|_1$ is the induced norm of $\mathcal{G}$ as a map from l_∞ to l_∞. Using the YJBK parametrization of stabilizing compensators the set of achievable closed loop impulse responses is $\{H - UQV : Q \in l_1\}$, where H, U and V are in l_1 and are determined by $\mathcal{G}$. Hence the l_∞ disturbance rejection problem is equivalent to

$$OPT: \qquad \inf\{\|H - K\|_1 : K \in K(U,V)\} =: \mu$$

where $K(U,V) := \{K \in l_1 : \exists Q \in l_1 \text{ satisfying } K = UQV\}$. This is the l_1 synthesis problem which was posed in [1], initially solved in [2] and generalized in [3] and other papers.

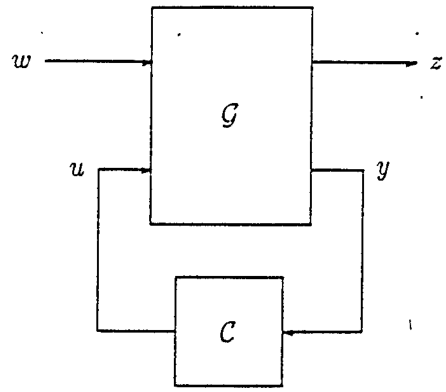

FIGURE 1: Standard Problem Setting

Introducing weighting systems in cascade with w and z broadens the class of specifications which a designer can address. Disturbances are thus generated by an input $\tilde{w}$ passed through a weight $\mathcal{W}_w$ and the regulated outputs are passed through a weight $\mathcal{W}_z$ to produce an output $\tilde{z}$. If W_z and W_w are in l_1 and have left and right inverses, respectively, in l_1 then the following specification can be easily addressed:

Cascade Weighted Specification:

- $w = \mathcal{W}_w \tilde{w}$ for some $\tilde{w} \in l_\infty$ with $\|\tilde{w}\|_\infty \leq 1$ implies $\mathcal{W}_z z \in l_\infty$ and $\|\mathcal{W}_z z\|_\infty \leq 1$.

Under the assumptions on W_z and W_w the cascade weighted disturbance rejection problem is equivalent to

$$\inf \{\|W_z H W_w - K\|_1 : K \in K(W_z U, V W_w)\}$$

and is hence an l_1 synthesis problem again.

$\mathcal{W}_z$ can be chosen to reflect an appealing class of criteria on the regulated output because of the definition of $\|\cdot\|_\infty$. For example, if $\hat{W}_z = [\ 1 \ \ 1 - z\ ]^T$ then $\|W_z z\|_\infty \leq 1$ if and only if $|z(k)| \leq 1$ and $|z(k+1) - z(k)| \leq 1$ for all k. Hence meeting the specification ensures a magnitude bound on both z and its rate of change. Additional n-th order differences of z for any desired n can be bounded by adding appropriate components to $\hat{W}_z$. Such specifications are of practical interest; bounding first-order differences ensures limited slew rates, and bounding second-order differences ensures limited accelerations.

The interpretation of the disturbance class generated by $\mathcal{W}_w$ is problematic, however. It is not known how to choose $\mathcal{W}_w$, for example, to produce a class of magnitude and rate bounded disturbances. Moreover, the meaning of choosing $\mathcal{W}_w$ based on its frequency response is unclear; the entire l_∞ design theory is aimed at time domain specifications.

Motivated by the appealing interpretation of $\mathcal{W}_z$, we will consider a weighted disturbance rejection problem aimed at satisfying a specification of the form

Weighted Specification:

- $\mathcal{W}_w w \in l_\infty$ and $\|\mathcal{W}_w w\|_\infty \leq 1$ implies $\mathcal{W}_z z \in l_\infty$ and $\|\mathcal{W}_z z\|_\infty \leq 1$.

A related design problem can be posed which is aimed at satisfying

Incremental Weighted Specification:

- $\|\mathcal{P}_n \mathcal{W}_w w\|_\infty \leq 1$ implies $\|\mathcal{P}_n \mathcal{W}_z z\|_\infty \leq 1$ *for all* n.

where $\mathcal{P}_n$ denotes truncation at time n. This is similar to the weighted specification and has a practical interpretation. It requires that the weighted error satisfy a constraint *up until any given time* provided that the weighted disturbance satisfies a constraint *up until the same time.* Note that any truncation of a right-supported signal is in l_∞ and hence any such signal constitutes a potential disturbance. We will see that the incremental specification is in general the more difficult to satisfy. Design to meet similar specifications has been considered previously in [4] and [5].

The remainder of the paper is organized as follows. Section 2 contains some background on the l_1 synthesis problem which will be required in later sections. In Section 3, the analysis problem of determining if a given system satisfies the above specifications is solved. For each specification, an appropriate system gain, which is also a norm, is defined and a method given for its computation. In Section 4, the synthesis problem of choosing C to minimize the desired norm is formulated, and for either norm is shown to be very similar to a standard l_1 synthesis problem. Section 5 contains some observations and conclusions.

Some statements of results are somewhat simplified, some proofs are omitted, and methods for the solution of optimization problems which arise are not given. In all cases, detailed results, proofs, and solution methods can be found in [6].

2 l_1 Synthesis

We will need some basic facts about the l_1 synthesis problem OPT. The crucial feature of OPT is that, under mild assumptions, it is equivalent to an infinite linear program. In particular, the following condition or something similar must be assumed [6].

Condition 2.1 *U and V have decompositions of the form $U = U_L \Sigma_U U_R$ and $V = V_L \Sigma_V V_R$ where*

- *$\Sigma_U \in l_1$, $\Sigma_V \in l_1$ are diagonal and nonsingular,*

- *neither* $(\hat{\Sigma}_U)_{ii}$ *nor* $(\hat{\Sigma}_V)_{jj}$ *have any zeros on the unit circle for any* i *or* j,
- $U_L, V_L \in l_1$ *are left invertible in* l_1, *and* $U_R, V_R \in l_1$ *are right invertible in* l_1.

If Condition 2.1 is satisfied then Bezout equations

$$\begin{bmatrix} U_L^{-L} \\ U_L^{\perp} \end{bmatrix} \begin{bmatrix} U_L & U_L^c \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad \begin{bmatrix} V_R \\ V_R^c \end{bmatrix} \begin{bmatrix} V_R^{-R} & V_R^{\perp} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{1}$$

can be constructed where all matrices are in l_1 and the feasible subspace $K(U,V)$ of OPT characterized as follows.

Fact 2.2 $K \in K$ *if and only if* $K \in l_1$ *and satisfies*

1. $\begin{bmatrix} U_L^{-L} \\ U_L^{\perp} \end{bmatrix} K \begin{bmatrix} V_R^{-R} & V_R^{\perp} \end{bmatrix} = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$
   where $*$ *denotes an irrelevant block.*
2. *For each* i *and* j, $(\hat{U}_L^{-L}\hat{K}\hat{V}_R^{-R})_{ij}$ *has all zeros of* $(\hat{\Sigma}_U)_{ii}(\hat{\Sigma}_V)_{jj}$ *in* D, *including multiplicities.*

Using Fact 2.2, OPT can be formulated as an infinite linear program whose variables are the closed loop impulse response elements; condition 1 imposes an infinite set of linear equality constraints (*convolution* constraints), and condition 2 a finite set of linear equality constraints (*interpolation* constraints).

3 Weighted Performance Analysis

In Section 3.1 we define a notion of stability and a norm on the stable systems appropriate to the incremental weighted specification. In Section 3.2 we do the same for the weighted specification and show, in addition, that the norm is an induced norm between weighted versions of l_∞. In both cases computation of the norm is similar to l_1 synthesis.

Throughout this section $\mathcal{H}$ is a given system. $\mathcal{W}_o$ and $\mathcal{W}_i$ are given weights; W_o and W_i are assumed to be in l_1 and to have left inverses in l_1. Hence the following Bezout equations can be constructed

$$\begin{bmatrix} W_o^{-L} \\ W_o^{\perp} \end{bmatrix} \begin{bmatrix} W_o & W_o^c \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad \begin{bmatrix} W_i^{-L} \\ W_i^{\perp} \end{bmatrix} \begin{bmatrix} W_i & W_i^c \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2}$$

where all matrices are in l_1. The additional symbols on the left hand sides of equations (2) denote *arbitrary* choices satisfying the equations.

3.1 Incremental Weighted Performance

Definition 3.1 $\mathcal{H}$ *is* incrementally stable w.r.t. $\mathcal{W}_o$, $\mathcal{W}_i$ *if*

$$\sup\{\|\mathcal{P}_n\mathcal{W}_o\mathcal{H}x\|_\infty : \mathcal{P}_n\mathcal{W}_i x \in l_\infty,\ \|\mathcal{P}_n\mathcal{W}_i x\|_\infty \le 1,\ n \in Z_+\} =: \rho_i(\mathcal{H};\mathcal{W}_o,\mathcal{W}_i) < \infty$$

$\rho_i(\mathcal{H};\mathcal{W}_o,\mathcal{W}_i)$ *is the* incremental gain of $\mathcal{H}$ w.r.t. $\mathcal{W}_o$, $\mathcal{W}_i$.

Under Definition 3.1, and for fixed weights, an incrementally stable system satisfies the incremental weighted specification if and only if its incremental gain is less than or equal to one. The next proposition shows that under our assumptions on the weights a system is incrementally stable if and only if its impulse response is in l_1. Moreover, the incremental gain is a norm on the incrementally stable systems.

Proposition 3.2 *$\mathcal{H}$ is incrementally stable w.r.t. $\mathcal{W}_o$, $\mathcal{W}_i$ if and only if $H \in l_1$. $\|\cdot\|^i_{\mathcal{W}_o,\mathcal{W}_i} := \rho_i(\cdot;\mathcal{W}_o,\mathcal{W}_i)$ is a norm on the incrementally stable systems.*

Proof: For the first sentence, if $H \in l_1$ then $W_o H W_i^{-L} \in l_1$ and for all x and n

$$\begin{aligned}\|\mathcal{P}_n\mathcal{W}_o\mathcal{H}x\|_\infty &= \|\mathcal{P}_n\mathcal{W}_o\mathcal{H}\mathcal{W}_i^{-L}\mathcal{W}_i x\|_\infty = \|\mathcal{P}_n\mathcal{W}_o\mathcal{H}\mathcal{W}_i^{-L}\mathcal{P}_n\mathcal{W}_i x\|_\infty \\ &\le \|W_o H W_i^{-L}\|_1 \|\mathcal{P}_n\mathcal{W}_i x\|_\infty.\end{aligned}$$

Hence $\rho_i(\mathcal{H};\mathcal{W}_o,\mathcal{W}_i) \le \|W_o H W_i^{-L}\|_1$. Conversely, if $H \notin l_1$ then $W_o H W_i^{-L} \notin l_1$ and hence, given any $c < \infty$, there exists $\bar{x} \in l_\infty$ such that $\|\mathcal{W}_o\mathcal{H}\mathcal{W}_i^{-L}\bar{x}\|_\infty > c\|W_i W_i^{-L}\|_1\|\bar{x}\|_\infty$. If we define $x := \mathcal{W}_i^{-L}\bar{x}$ then

$$\sup\{\|\mathcal{P}_n\mathcal{W}_o\mathcal{H}x\|_\infty : n \in Z_+\} = \|\mathcal{W}_o\mathcal{H}x\|_\infty = \|\mathcal{W}_o\mathcal{H}\mathcal{W}_i^{-L}\bar{x}\|_\infty > c\|W_i W_i^{-L}\|_1\|\bar{x}\|_\infty.$$

Hence there exists $n \in Z_+$ such that

$$\|\mathcal{P}_n\mathcal{W}_o\mathcal{H}x\|_\infty > c\|W_i W_i^{-L}\|_1\|\bar{x}\|_\infty \ge c\|\mathcal{W}_i x\|_\infty \ge c\|\mathcal{P}_n\mathcal{W}_i x\|_\infty$$

so $\rho_i(\mathcal{H};\mathcal{W}_o,\mathcal{W}_i) > c$.

The second sentence is easily verified. □

The incremental gain of a given system w.r.t. given weights can be computed by solving an l_1 synthesis problem with a special form, as the next theorem shows. Its proof is omitted in the interest of brevity.

Theorem 3.3 $\|H\|^i_{W_o,W_i} = \inf\left\{\|W_o H W_i^{-L} - K\|_1 : K \in K(I, W_i^\perp)\right\}$.

It is easy to see that I and $W_i^\perp$ satisfy Condition 2.1; the obvious decompositions are: $U_L = \Sigma_U = U_R = I$ and $V_L = \Sigma_V = I$, $V_R = W_i^\perp$. Hence $\|H\|^i_{W_o,W_i}$ can be computed using l_1 synthesis techniques.

3.2 Weighted Performance

Definition 3.4 $\mathcal{H}$ *is* stable w.r.t. $\mathcal{W}_o$, $\mathcal{W}_i$ *if*

$$\sup \{\|\mathcal{W}_o \mathcal{H} x\|_\infty : \mathcal{W}_i x \in l_\infty,\ \|\mathcal{W}_i x\|_\infty \leq 1\} =: \rho(\mathcal{H}; \mathcal{W}_o, \mathcal{W}_i) < \infty$$

$\rho(\mathcal{H}; \mathcal{W}_o, \mathcal{W}_i)$ *is the* gain of $\mathcal{H}$ w.r.t. $\mathcal{W}_o$, $\mathcal{W}_i$.

Under Definition 3.4, and for fixed weights, a stable system satisfies the weighted specification if and only if its gain is less than or equal to one. The next proposition shows that under our assumptions on weights a system is stable if and only if its impulse response is in l_1. Moreover, the gain is a norm on the stable systems.

Proposition 3.5 *$\mathcal{H}$ is stable w.r.t. $\mathcal{W}_o$, $\mathcal{W}_i$ if and only if $H \in l_1$. $\|\cdot\|_{\mathcal{W}_o,\mathcal{W}_i} := \rho(\cdot; \mathcal{W}_o, \mathcal{W}_i)$ is a norm on the stable systems.*

Proof: For the first sentence, if $H \in l_1$ then $W_o H W_i^{-L} \in l_1$ and for all x

$$\|\mathcal{W}_o \mathcal{H} x\|_\infty = \|\mathcal{W}_o \mathcal{H} \mathcal{W}_i^{-L} \mathcal{W}_i x\|_\infty \leq \|W_o H W_i^{-L}\|_1 \|\mathcal{W}_i x\|_\infty.$$

Hence $\rho(\mathcal{H}; \mathcal{W}_o, \mathcal{W}_i) \leq \|W_o H W_i^{-L}\|_1$. Conversely, if $H \notin l_1$ then $W_o H W_i^{-L} \notin l_1$ and hence, given any $c < \infty$, there exists $\tilde{x} \in l_\infty$ such that $\|\mathcal{W}_o \mathcal{H} \mathcal{W}_i^{-L} \tilde{x}\|_\infty > c\|W_i W_i^{-L}\|_1 \|\tilde{x}\|_\infty$. If we define $x := \mathcal{W}_i^{-L} \tilde{x}$ then $\|\mathcal{W}_o \mathcal{H} x\|_\infty > c\|W_i W_i^{-L}\|_1 \|\tilde{x}\|_\infty \geq \|\mathcal{W}_i x\|_\infty$ so $\rho(\mathcal{H}; \mathcal{W}_o, \mathcal{W}_i) > c$.

The second sentence is easily verified. □

The gain of a given system w.r.t. given weights can be computed by solving a problem similar to an l_1 synthesis problem, as the next theorem shows. Its proof is similar to that of Theorem 3.3 and is also omitted in the interest of brevity.

Theorem 3.6 $\|\mathcal{H}\|_{\mathcal{W}_o,\mathcal{W}_i} = \inf\left\{\|W_o H W_i^{-L} - K\|_1 : K \in K_Z(I, W_i^{\perp})\right\}$, *where $K_Z(\cdot,\cdot)$ is defined as $K(\cdot,\cdot)$ in Section 1 except that Q and K are allowed to range over $l_1(\mathrm{Z})$.*

As in the incremental problem of Section 3.1, I and $W_i^{\perp}$ satisfy Condition 2.1. As a result the feasible subspace $K_Z(I, W_i^{\perp})$ of the infimization in Theorem 3.6 has a characterization similar to that of $K(I, W_i^{\perp})$, and the computation of $\|\mathcal{H}\|_{\mathcal{W}_o,\mathcal{W}_i}$ is equivalent to an infinite linear program. Moreover, approximate solution methods analogous to those for standard l_1 exist.

The gain of a system w.r.t. given weights is in general smaller than its incremental gain w.r.t. the same weights, as the following simple example shows.

Example: Let $\hat{H} = \hat{W}_o = 1$,

$$\hat{W}_i = \begin{bmatrix} 1 \\ -1 + 3z \end{bmatrix}$$

and choose $\hat{W_{\mathrm{i}}}^{-L} = [\ 1 \quad 0\ ]$ and $\hat{W_{\mathrm{i}}}^{\perp} = [\ 1 - 3z \quad 1\ ]$ to satisfy the Bezout equations (2). $\|H\|^{i}_{W_{\mathrm{o}},W_{\mathrm{i}}}$ and $\|H\|_{W_{\mathrm{o}},W_{\mathrm{i}}}$ are computed by solving $\inf \|[\ 1 \quad 0\ ] + \hat{q}[\ 1 - 3z \quad 1\ ]\|_{A} =: \gamma$ where q ranges over l_1 and $l_1(\mathrm{Z})$, respectively. It is not hard to check that $\gamma \geq 1$ when q ranges over l_1 since

$$\|[\ 1 \quad 0\ ] + \hat{q}[\ 1 - 3z \quad 1\ ]\|_{A} = \|1 + \hat{q}(1 - 3z)\|_{A} + \|\hat{q}\|_{A} \geq (1 - |q_0|) + |q_0| = 1$$

where q_0 is the first element of q. On the other hand, if we take $\hat{q} = \frac{1}{3}z^{-1}$ then $q \in l_1(\mathrm{Z})$ and

$$\|[\ 1 \quad 0\ ] + \hat{q}[\ 1 - 3z \quad 1\ ]\|_{A} = \|1 + \hat{q}(1 - 3z)\|_{A} + \|\hat{q}\|_{A} = \frac{2}{3}.$$

Hence $\gamma \leq \frac{2}{3}$ if q is allowed to range over $l_1(\mathrm{Z})$.

Next we will show that the gain of a system w.r.t. given weights is an induced norm between weighted versions of l_∞.

Definition 3.7 *If x is any signal, $\mathcal{W}$ is any system, and $\mathcal{W}x \in l_\infty$ then $\rho_{\mathcal{W}}(x) := \|\mathcal{W}x\|_\infty$ is the* $\mathcal{W}$-weighted l_∞-norm *of x.*

With no assumptions on $\mathcal{W}$, $\rho_{\mathcal{W}}(\cdot)$ is actually only a semi-norm, as it can have a null space. Moreover, it need not be defined on all of l_∞ and can be defined for signals not in l_∞. Under our assumptions on weights, however, it is defined precisely on l_∞ and is a norm.

Proposition 3.8 *If $W \in l_1$ and has a left inverse in l_1 then $\mathcal{W}x \in l_\infty$ if and only if $x \in l_\infty$, and $\|\cdot\|_{\mathcal{W}} := \rho_{\mathcal{W}}(\cdot)$ is a norm on l_∞.*

Proof: Let W^{-L} denote any left inverse of W in l_1. For the first sentence, if $x \in l_\infty$ then $\mathcal{W}x \in l_\infty$ since $W \in l_1$. Conversely, if $\mathcal{W}x \in l_\infty$ then $x = \mathcal{W}^{-L}\mathcal{W}x \in l_\infty$ since $W^{-L} \in l_1$. For the second sentence, the properties of a semi-norm follow from the linearity of $\mathcal{W}$ and the corresponding properties of $\|\cdot\|_\infty$. Moreover, $\|x\|_{\mathcal{W}} = 0 \Rightarrow \mathcal{W}x = 0 \Rightarrow x = \mathcal{W}^{-L}\mathcal{W}x = 0$. □

l_∞ under $\|\cdot\|_{\mathcal{W}}$ can be called *$\mathcal{W}$-weighted l_∞*. It is clear that $\|\mathcal{H}\|_{\mathcal{W}_{\mathrm{o}},\mathcal{W}_{\mathrm{i}}}$ is the induced norm of $\mathcal{H}$ viewed as a map from $\mathcal{W}_{\mathrm{i}}$-weighted l_∞ to $\mathcal{W}_{\mathrm{o}}$-weighted l_∞ since

$$\begin{aligned} \|\mathcal{H}\|_{\mathcal{W}_{\mathrm{o}},\mathcal{W}_{\mathrm{i}}} &= \sup\{\|\mathcal{W}_{\mathrm{o}}\mathcal{H}x\|_\infty : \mathcal{W}_{\mathrm{i}}x \in l_\infty,\ \|\mathcal{W}_{\mathrm{i}}x\|_\infty \leq 1\} \\ &= \sup\left\{\|\mathcal{H}x\|_{\mathcal{W}_{\mathrm{o}}} : x \in l_\infty,\ \|x\|_{\mathcal{W}_{\mathrm{i}}} \leq 1\right\}. \end{aligned}$$

using the definition of $\|\mathcal{H}\|_{\mathcal{W}_{\mathrm{o}},\mathcal{W}_{\mathrm{i}}}$ for the first equality and Proposition 3.8 for the second.

4 Synthesis

In this section, we show that the problems of minimizing the incremental gain and the gain, respectively, of the closed loop system in Figure 1 can be formulated as infinite linear programs in a manner similar to the l_1 synthesis problem. Section 4.1 considers the incremental gain and Secton 4.2 the gain.

It is assumed that the U and V matrices obtained from $\mathcal{G}$ via the YJBK parametrization satisfy Condition 2.1. W_z and W_w are assumed to be in l_1 and to have left inverses in l_1. The associated Bezout equations are

$$\begin{bmatrix} W_z^{-L} \\ W_z^{\perp} \end{bmatrix} \begin{bmatrix} W_z & W_z^c \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad \begin{bmatrix} W_w^{-L} \\ W_w^{\perp} \end{bmatrix} \begin{bmatrix} W_w & W_w^c \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{3}$$

and the additional symbols on the left hand sides denote *arbitrary* choices satisfying the equations.

4.1 Incremental Weighted Synthesis

The YJBK parametrization and Theorem 3.3 can be combined to formulate the incremental weighted synthesis problem as follows.

$$OPT_i: \qquad \inf\left\{\|W_z H W_w^{-L} - K\|_1 : K \in K_i\right\} =: \mu_i$$

where $K_i := \left\{K \in l_1 : \exists Q_C, Q_W \in l_1 \text{ satisfying } K = W_z U Q_C V W_w^{-L} + Q_W W_w^{\perp}\right\}$. The parameter Q_C corresponds to stabilizing compensators and Q_W to computing the closed loop incremental gain corresponding to each. K_i is related to the feasible subspace $K(\cdot,\cdot)$ of an l_1 synthesis problem.

Lemma 4.1 *$K \in K_i$ if and only if $K \in l_1$ and $KW_w \in K(W_z U, V)$.*

Proof: If $K \in K_i$ then $K \in l_1$ and $K = W_z U Q_C V W_w^{-L} + Q_W W_w^{\perp}$. Hence, using (3), $KW_w = W_z U Q_C V$. Since $Q_C \in l_1$, $KW_w \in K(W_z U, V)$. Conversely, if $KW_w \in K(W_z U, V)$ then $K \in l_1$ and $KW_w = W_z U Q_C V$ for some $Q_C \in l_1$. Using the reverse of (3),

$$K = K(W_w W_w^{-L} + W_w^c W_w^{\perp}) = W_z U Q_C V W_w^{-L} + K W_w^c W_w^{\perp}$$

where $Q_W := K W_w^c \in l_1$ since both K and W_w^c are. Hence $K \in K_i$. □

Using Lemma 4.1 we can easily establish a characterization of K_i similar to that of $K(\cdot,\cdot)$ given in Fact 2.2.

Theorem 4.2 *$K \in K_i$ if and only if $K \in l_1$ and satisfies*

1. $\begin{bmatrix} U_L^{-L}W_z^{-L} \\ U_L^{\perp}W_z^{-L} \\ W_z^{\perp} \end{bmatrix} K \begin{bmatrix} W_wV_R^{-R} & W_wV_R^{\perp} \end{bmatrix} = \begin{bmatrix} * & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

   where $*$ denotes an irrelevant block.

2. *For each i and j, $(\hat{U}_L^{-L}\hat{W}_z^{-L}\hat{K}\hat{W}_w\hat{V}_R^{-R})_{ij}$ has all zeros of $(\hat{\Sigma}_U)_{ii}(\hat{\Sigma}_V)_{jj}$ in D, including multiplicities.*

Proof: U has a decomposition $U = U_L\Sigma_U U_R$ as in Condition 2.1. It is easy to see that, since W_z is left invertible in l_1, $W_zU = (W_zU_L)\Sigma_U U_R$ is a decomposition of W_zU of the same form. Hence a Bezout equation

$$\begin{bmatrix} U_L^{-L}W_z^{-L} \\ U_L^{\perp}W_z^{-L} \\ W_z^{\perp} \end{bmatrix} \begin{bmatrix} W_zU_L & W_zU_L^c & W_z^c \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

for W_zU can be constructed using (1) and (3). Fact 2.2 can now be applied to characterize $K(W_zU, V)$ and the proof is completed by combining this characterization with Lemma 4.1. □

Theorem 4.2 allows OPT_i to be formulated as an infinite linear program. Condition 1 imposes an infinite set of convolution constraints and condition 2 a finite set of interpolation constraints on $W_zHW_w^{-L} - K$. Moreover, approximate solution methods analogous to those for the l_1 synthesis problem exist.

4.2 Weighted Synthesis

The YJBK parametrization and Theorem 3.6 can be combined to formulate the weighted synthesis problem as follows.

$$OPT_W : \qquad \inf\left\{\|W_zHW_w^{-L} - K\|_1 : K \in K_W\right\} =: \mu_W$$

where $K_W := \left\{K \in l_1(\mathrm{Z}) : \exists Q_C \in l_1,\ Q_W \in l_1(\mathrm{Z}) \text{ satisfying } K = W_zUQ_CVW_w^{-L} + Q_WW_w^{\perp}\right\}$. Q_C corresponds to stabilizing compensators and Q_W to computing the closed loop gain corresponding to each. K_W is related to the feasible subspace $K(\cdot,\cdot)$ of an l_1 synthesis problem in the same way that K_i is, and an analogous characterization can be obtained. The next two results show this and are presented without proof; the proofs of Lemma 4.1 and Theorem 4.2, respectively, apply with obvious modifications.

Lemma 4.3 *$K \in K_W$ if and only if $K \in l_1(\mathrm{Z})$ and $KW_w \in K(W_zU, V)$.*

Theorem 4.4 *$K \in K_W$ if and only if $K \in l_1(\mathrm{Z})$ and satisfies*

1. $$\begin{bmatrix} U_L^{-L} W_z^{-L} \\ U_L^{\perp} W_z^{-L} \\ W_z^{\perp} \end{bmatrix} K \begin{bmatrix} W_w V_R^{-R} & W_w V_R^{\perp} \end{bmatrix} = \begin{bmatrix} * & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   where $*$ denotes an irrelevant block.

2. *For each i and j, $(\hat{U}_L^{-L} \hat{W}_z^{-L} \hat{K} \hat{W}_w \hat{V}_R^{-R})_{ij}$ has all zeros of $(\hat{\Sigma}_U)_{ii}(\hat{\Sigma}_V)_{jj}$ in D, including multiplicities.*

Theorem 4.4 allows OPT_W to be formulated as an infinite linear program; condition 1 imposes an infinite set of convolution constraints and condition 2 a finite set of interpolation constraints on $W_z H W_w^{-L} - K$ as in the incremental synthesis problem. However in this case K and hence $W_z H W_w^{-L} - K$ ranges over $l_1(\mathrm{Z})$, requiring appropriate modifications to the approximate solution methods for OPT_I.

5 Conclusions

Weights are often used to increase the range of specifications which a designer can address. The simplest scheme is cascade weighting, but it is problematic in an l_∞ setting in that the disturbance class does not have a clear interpretation, while the error class does. It is interesting to note that this distinction does not arise for l_2 signals (it is not hard to show that rational H_∞ weights with no zeros on the unit circle can be replaced by their outer factors and hence inverted).

Both the weighted l_∞ performance specifications considered here measure disturbance and error as errors are measured in the cascade scheme. This has an appealing practical interpretation in that it allows incorporation of criteria in addition to disturbance and error magnitudes, e.g., rate and acceleration bounds. Such criteria cannot be addressed using other design methods.

Analysis and synthesis for each specification can be done by methods similar to standard l_1 synthesis and, in fact, may be simpler in some respects. In l_1 synthesis, for example, suboptimal compensators can be obtained by optimizing over achievable finitely supported closed loop impulse responses of a given length. As the length is increased, the performance of the suboptimal compensators approaches the optimal. When cascade weights with infinite impulse responses are introduced this method fails and the weights must be approximated as finite impulse response, leading to high order compensators. However, weighted specifications of the type considered here have their most appealing interpretations when the weights are finite impulse response (i.e., they measure rates, accelerations, etc.).

Because system norms have been defined appropriate to each specification, new problems of robustness with respect to classes of norm-bounded perturbations can be posed as well.

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